A PROPERTY OF NONSEPARATED CONVEX SETS[1]

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Introduction. The theory of convex sets, and more particularly their separation properties, is a very useful tool in the solution of many optimal control problems. In the case of linear problems the theory of convex sets gives an immediate answer, see for instance LaSalle [1] and Neustadt [2]. When the problem is nonlinear it is sometimes useful to study its linearization and to prove that some results obtained for this linearization are indeed valid for the nonlinear problem itself. In many instances this proof can be carried out by elementary topological arguments. Pontryagin et al, pp. 97 and 111, outline such a proof using the concept of topological index. In the present paper we prove a more general result of the same type using Brouwer's fixed point theorem. We shall prove that

THEOREM I. *Let K_1 and K_2 be two nonseparated convex sets in a Euclidean space E^n. We assume that $0 \in \overline{K_1} \cap \overline{K_2}$ and $0 \notin K_1 \cap K_2$. Let L be a positive constant. For every $\eta > 0$ we are given a continuous mapping ϕ_1^η from K_1 into E^n and a continuous mapping ϕ_2^η from K_2 into E^n. We assume that for every $\eta > 0$ and for $i = 1, 2$ we have*

$$(1) \qquad |\phi_i(e)^\eta - e| \le L|e|^2 + \eta \quad \text{for all } e \in K_i$$

where $|e|$ stands for the Euclidean length of the vector e. Then the set $\bigcup_{\eta>0}(\phi_1^\eta(K_1) \cap \phi_2(K_2))$ is not empty.

Notations and classical results. The scalar product of two vectors a and b is denoted by $a \cdot b$. If $a \in E^n$ and $\epsilon > 0$ we shall denote by $N(a, \epsilon)$ the ϵ-neighborhood of the point a. The smallest linear variety containing a subset A of E^n is denoted by $L(A)$. If A is a subset of E^n we shall denote by int A the interior of A with respect to E^n and by rint A the interior of A with respect to $L(A)$. Two subsets A and B of E^n are separated if there exists a vector $p \in E^n$ and a real number α such that

$$(2) \qquad x \cdot p \geqq \alpha \quad \text{for all } x \in A,$$

$$(3) \qquad x \cdot p \leqq \alpha \quad \text{for all } x \in B.$$

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We recall the classical

SEPARATION THEOREM. *If two convex subsets A and B of E^n are not separated then $L(A\cup B)=E^n$ and there exists a point x which belongs to* rint A *and to* rint B.

Before proving Theorem I we shall state and prove Lemma I which is a particular case of Theorem I. (One of the given convex sets is merely a one-dimensional linear segment and the mapping corresponding to this linear segment is the identity mapping.)

LEMMA I. *Let K be a convex set in the Euclidean space E^n. We assume that the origin 0 belongs to $\overline{K}$ and that there exists a point x interior to the set K. Let L be a positive constant. For every $\eta>0$ we are given a continuous mapping ϕ^η from K into E^n. We assume that for every $\eta>0$ we have*

$$|\phi^\eta(e)-e|\leqq L|e|^2+\eta \quad \text{for all } e\in K. \tag{4}$$

Then there is a $\delta>0$, an $\alpha>0$ and a point y interior to the set K such that

$$\phi^\delta(y)=\alpha x. \tag{5}$$

PROOF OF LEMMA I. Since $x\in\text{int } K$ we know that there exists an $\epsilon>0$ with $N(x,\epsilon)\subset K$. Let α and δ be positive constants such that

$$L\alpha^2(|x|+\epsilon)^2+\delta\leqq\frac{\alpha\epsilon}{4} \tag{6}$$

and

$$\alpha\leqq 1. \tag{7}$$

Since the set K is convex, $0\in\overline{K}$, and $N(x,\epsilon)\subset K$ we have then immediately $N(\alpha x,\alpha\epsilon/2)\subset K$. Let $h(e)=e-\phi^\delta(e)+\alpha x$. The function h is continuous and maps $N(\alpha x,\alpha\epsilon/2)$ into itself since

$$|h(e)-\alpha x|=|e-\phi^\delta(e)|\leqq L\alpha^2(|x|+\epsilon)^2+\delta\leqq\frac{\alpha\epsilon}{4} \tag{8}$$

for all $e\in N(\alpha x,\alpha\epsilon/2)$. Hence, by Brouwer's Fixed Point Theorem, there is a $y\in N(\alpha x,\alpha\epsilon/2)$ such that $h(y)=y$, i.e., $\phi^\delta(y)=\alpha x$. We have $y\in\text{int } K$ and Lemma I is proved.

PROOF OF THEOREM I. From the Separation Theorem we know that there exists a point e^* such that $e^*\in\text{rint } K_i$, $i=1,2$. We have $e^*\neq 0$ since $0\notin K_1\cap K_2$. We shall first prove that there exists an integer k with $1\leqq k\leqq n$, $n+1$ vectors $e_1, e_2, \cdots, e_{n+1}$ and $n+1$ positive numbers $\lambda_1, \lambda_2, \cdots, \lambda_{n+1}$ such that

(i) $e_i \in K_1$ for $i=1, \cdots, k$; $e^* = \sum_{i=1}^{k} \lambda_i e_i$ and $\sum_{i=1}^{k} \lambda_i < 1$,

(ii) $e_i \in K_2$ for $i=k+1, \cdots, n+1$; $e^* = \sum_{i=k+1}^{n+1} \lambda_i e_i$ and $\sum_{i=k+1}^{n+1} \lambda_i < 1$,

(iii) any n vectors among the $n+1$ vectors $e_1, e_2, \cdots, e_{n+1}$ are linearly independent,

(iv) $e_i \cdot e_j > 0$ for $i, j = 1, 2, \cdots, n+1$,

(v) the sets A_1 and A_2 defined by

$$A_1 = \left\{ \sum_{i=1}^{k} \mu_i e_i : \sum_{i=1}^{k} \mu_i < 1, \mu_j > 0 \quad \text{for } j = 1, 2, \cdots, k \right\}, \tag{9}$$

$$A_2 = \left\{ \sum_{i=k+1}^{n+1} \mu_i e_i : \sum_{i=k+1}^{n+1} \mu_i < 1, \mu_j > 0 \quad \text{for } j = k+1, \cdots, n+1 \right\} \tag{10}$$

are not separated.

From the Separation Theorem we know that $L(K_1 \cup K_2) = E^n$. Hence there exists n linearly independent vectors $a_1, a_2, \cdots, a_n$ in $K_1 \cup K_2$. We may always assume that e^* is one of the n vectors $a_1, a_2, \cdots, a_n$ and for some integer k with $1 \leq k \leq n$ we have

$$a_i \in K_1 \quad \text{for} \quad i = 1, 2, \cdots, k,$$
$$a_k = e^*,$$
$$a_i = K_2 \quad \text{for} \quad i = k, k+1, \cdots, n.$$

Let $P_1 = L(\{0, a_1, a_2, \cdots, a_k\})$, $P_2 = L(\{0, a_k, a_{k+1}, \cdots, a_n\})$, $K_i^* = K_i \cap P_i$, $i=1, 2$. By construction we have $e^* \in \text{rint } K_1^*$, $i=1, 2$ and $L(K_1^* \cup K_2^*) = E^n$. Hence, by the Separation Theorem, the convex sets K_1^* and K_2^* are nonseparated. Since $e^* \in \text{rint } K_1^*$ then there exist k linearly independent vectors $e_1, e_2, \cdots, e_k$ in K_1^* such that $e^* \in \text{rint } A_1$ where A_1 is defined as above. Similarly since $e^* \in \text{rint } K_2^*$ then there exists $n-k+1$ linearly independent vectors $e_{k+1}, \cdots, e_{n+1}$ in K_2^* such that $e^* \in \text{rint } A_2$ where A_2 is defined as above. By construction any n vectors among the $n+1$ vectors $e_1, e_2, \cdots, e_{n+1}$ are linearly independent and $L(A_1 \cup A_2) = E^n$. Hence, from the Separation Theorem, we know that the sets A_1 and A_2 are nonseparated. There is no loss of generality by assuming that $e_i \cdot e_j > 0$ for all i and $j = 1, 2, \cdots, n+1$. Indeed for any $\rho \in (0, 1)$ let $e_i(\rho) = e^* + \rho(e_i - e^*)$. For a small enough ρ we have $e_i(\rho) \cdot e_j(\rho) > 0$ for all i and $j = 1, 2, \cdots, n+1$ and none of the other properties are violated. Since $e^* \in \text{rint } A_1$ and $e^* \in \text{rint } A_2$ then there exist $n+1$ positive numbers $\lambda_1, \lambda_2, \cdots, \lambda_{n+1}$ satisfying the conditions stated above. This concludes the proof of relations (i) through (v).

By construction we have $A_i \subset K_i$, $i=1, 2$. From now to the end of the proof we shall restrict our attention to the sets A_1 and A_2 and prove that

$$\bigcup_{\eta>0} (\phi_1^\eta(A_1) \cap \phi_2^\eta(A_2))$$

is not empty which a fortiori implies that

$$\bigcup_{\eta>0} (\phi_1^\eta(K_1) \cap \phi_2^\eta(K_2))$$

is not empty.

Let $x_1, x_2, \cdots, x_{n+1}$ be vectors in E^{n+1} determined by

$$x_i = \left(e_i, \frac{1}{k\lambda_i}\right) \qquad \text{for } i = 1, 2, \cdots, k \tag{11}$$

$$= \left(-e_i, \frac{1}{(n+1-k)\lambda_i}\right) \qquad \text{for } i = k+1, \cdots, n+1. \tag{12}$$

It is easy to prove that the vectors $x_1, x_2, \cdots, x_{n+1}$ are linearly independent and that

$$\sum_{i=1}^{n+1} \lambda_i x_i = (0, 0, \cdots, 0, 2). \tag{13}$$

Let A be the subset of E^{n+1} defined by

$$A = \left\{ \sum_{i=1}^{n+1} \mu_i x_i \colon \sum_{i=1}^{n+1} \mu_i < 1, \mu_j > 0 \qquad \text{for } j = 1, 2, \cdots, n+1 \right\}$$

and let $x^* = (0, 0, \cdots, 0, 1)$. We have $x^* \in \operatorname{int} A$.

For any $\eta > 0$ let ϕ^η be a continuous mapping from A into E^{n+1} defined by

$$\begin{aligned} \phi^\eta\left(\sum_{i=1}^{n+1} \mu_i x_i\right) &= \left(\phi_1^\eta\left(\sum_{i=1}^{k} \mu_i e_i\right), \sum_{i=1}^{k} \frac{\mu_i}{k\lambda_i}\right) \\ &\quad + \left(-\phi_2^\eta\left(\sum_{i=k+1}^{n+1} \mu_i e_i\right), \sum_{i=k+1}^{n+1} \frac{\mu_i}{(n+1-k)\lambda_i}\right). \end{aligned} \tag{14}$$

For every $\eta > 0$ the mapping ϕ^η is well defined since the representation $\sum_{i=1}^{n+1} \mu_i x_i$ is unique (the vectors $x_1, x_2, \cdots, x_{n+1}$ are linearly independent) and since $\sum_{i=1}^{k} \mu_i e_i \in A_1$ and $\sum_{i=k+1}^{n-1} \mu_i e_i \in A_2$.

We have

$$\phi^{\eta}\left(\sum_{i=1}^{n+1}\mu_i x_i\right)-\sum_{i=1}^{n+1}\mu_i x_i=\left(\left(\phi_1^{\eta}\left(\sum_{i=1}^{k}\mu_i e_i\right)-\sum_{i=1}^{k}\mu_i e_i\right)\right.$$
$$\left.-\left(\phi_2^{\eta}\left(\sum_{i=k+1}^{n+1}\mu_i e_i\right)-\sum_{i=k+1}^{n+1}\mu_i e_i\right),0\right).$$

Hence

$$\begin{aligned}&\left|\phi^{\eta}\left(\sum_{i=1}^{n+1}\mu_i x_i\right)-\sum_{i=1}^{n+1}\mu_i x_i\right|\\ &\leqq\left|\phi_1^{\eta}\left(\sum_{i=1}^{k}\mu_i e_i\right)-\sum_{i=1}^{k}\mu_i e_i\right|+\left|\phi_2^{\eta}\left(\sum_{i=k+1}^{n+1}\mu_i e_i\right)-\sum_{i=k+1}^{n+1}\mu_i e_i\right|\\ &\leqq L\left|\sum_{i=1}^{k}\mu_i e_i\right|^2+\eta+L\left|\sum_{i=k+1}^{n+1}\mu_i e_i\right|^2+\eta\\ &\leqq L\max_{j=1,\cdots,n+1}|e_j|^2\sum_{i=1}^{n+1}|\mu_i|^2+2\eta.\end{aligned}\tag{15}$$

Since the vectors $x_1, x_2, \cdots, x_{n+1}$ are linearly independent there exists a constant $N<+\infty$ such that

$$\sum_{i=1}^{n+1}|\mu_i|^2\leqq N\left|\sum_{i=1}^{n+1}\mu_i x_i\right|^2\tag{16}$$

for all $\mu_1, \mu_2, \cdots, u_{n+1}$.

From relations (15) and (16) we obtain

$$\left|\phi\left(\sum_{i=1}^{n+1}\mu_i x_i\right)-\sum_{i=1}^{n+1}\mu_i x_i\right|\leqq L^*\left|\sum_{i=1}^{n+1}\mu_i x_i\right|^2+2\eta\tag{17}$$

where

$$L^*=L\max_{j=1,\cdots,n+1}|e_j|^2N.$$

The relation (17) can be written

$$|\phi(e)-e|\leqq L^*|e|^2+2\eta\qquad\text{for all } e\in A.$$

By Lemma I there is an $\bar{x}\in\operatorname{int}A$, a $\delta>0$ and an $\alpha>0$ such that

$$\phi^{\delta}(\bar{x})=\alpha x^*.\tag{18}$$

We have then $\bar{x}=\sum_{i=1}^{n+1}\nu_i x_i$ for some $\nu_i, \nu_2, \cdots, \nu_{n+1}$ such that $\sum_{i=1}^{n+1}\nu_i<+1$ and $\nu_j>0$ for $j=1, 2, \cdots, n+1$. This implies that

$$(0, 0, \cdots, 0, \alpha) = \phi^\delta(\tilde{x}) = \phi^\delta\left(\sum_{i=1}^{n+1} \nu_i x_i\right)$$
$$= \left(\phi_1^\delta\left(\sum_{i=1}^{k} \nu_i e_i\right), \sum_{i=1}^{k} \frac{\nu_i}{k\lambda_i}\right)$$
$$+ \left(-\phi_2^\delta\left(\sum_{i=k+1}^{n+1} \nu_i e_i\right), \sum_{i=k+1}^{n+1} \frac{\nu_i}{(n+1-k)\lambda_i}\right)$$

and in particular that

$$\phi_1^\delta\left(\sum_{i=1}^{k} \nu_i e_i\right) - \phi_2^\delta\left(\sum_{i=k+1}^{n+1} \nu_i e_i\right) = 0. \tag{19}$$

Let

$$e^1 = \sum_{i=1}^{k} \nu_i e_i \qquad \text{and} \qquad e^2 = \sum_{i=k+1}^{n+1} \nu_i e_i.$$

We have then $\phi_1^\delta(e^1) = \phi_2^\delta(e^2)$ with $e^1 \in A_1$ and $e^2 \in A_2$. This concludes the proof of Theorem I.

FINAL REMARKS. The following particular cases of Lemma I and Theorem I are sometimes sufficient in the theory of optimum control.

LEMMA I*. *Let K be a convex set in the Euclidean space E^n. We assume that the origin 0 belongs to $\overline{K}$ and that there exists a point x interior to the set K. Let L be a positive constant. We are given a continuous mapping ϕ from K into E^n such that*

$$|\phi(e) - e| \leqq L|e|^2 \qquad \text{for all } e \in K. \tag{20}$$

Then there is an $\alpha > 0$ and a point y interior to the set K such that

$$\phi(y) = \alpha x. \tag{21}$$

THEOREM I*. *Let K_1 and K_2 be two nonseparated convex sets in an Euclidean space E^n. We assume that $0 \in \overline{K}_1 \cap \overline{K}_2$ and $0 \notin K_1 \cap K_2$. Let L be a positive constant. We are given a continuous mapping ϕ_1 from K_1 into E^n and a continuous mapping ϕ_2 from K_2 into E^n. We assume that for $i = 1, 2$ we have*

$$|\phi_i(e) - e| \leqq L|e|^2 \qquad \text{for all } e \in K_i. \tag{22}$$

Then the set $\phi_1(K_1) \cap \phi_2(K_2)$ is not empty.

Acknowledgment. An independent proof of Lemma I* was given in Halkin [3]. This earlier proof was due to G. Stephen Jones and

should be considered as the inspiring force of the proof of the more general result presented in the present paper.

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