A CHARACTERIZATION OF UNIONS OF TWO STAR-SHAPED SETS

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We shall use the notation $S(x)$ to denote the star of x in M, that is, the set of all points $y \in M$ such that the segment xy is contained in M. The star (in M) of a set A is defined by $S(A) = \bigcup_{x \in A \cap M} S(x)$. We use the terminology "the point x sees the point y" to mean the closed segment xy is contained in M.

Let M be a closed set in E_r. Suppose there exists a line segment S_1 such that each triple of points x, y, z in M determines at least one point $p \in S_1$ such that at least two of the points x, y, z see the point p via M. Valentine [1, Problem 6.6, p. 178] has conjectured that this property characterizes M as the union of at most two star-shaped sets. That this condition is necessary follows immediately by choosing S_1 to be any line segment which intersects the kernels of the two star-shaped sets. A further property which is enjoyed by every union of two star-shaped sets is the following.

CONDITION A. If $S_1 = \bigcup_{i=1}^{m} I_i$ where the I_i are closed intervals with at most end points in common, then of the intervals I_i there is at least one pair (say I_r and I_s) such that at least two of every triple of points of M, see a common point of $I_r \cup I_s$ via M.

If it is true that of each triple of points of M at least two of them see a common point of a single interval (say I_r), then the pair I_r and I_s where $I_s = I_r$ satisfies the conclusion of condition A. The reader will note that if $m = 2$, then Condition A implies Valentine's property.

Next we note that a set M satisfying either Valentine's property or Condition A consists of at most two components, for if M had as many as three components the selection of a point from each of the components would violate Valentine's condition. In the two component case Valentine's condition can be stated as follows. If x and y are in the same component of M, then there exists a point $p \in S_1$ such that the segments xp and yp are contained in M. But then an application of a generalization of Krasnosel'skiĭ's theorem [1, Theorem 6.18, p. 85] tells us that each component is star-shaped. Before proceeding with the case in which M is connected we prove the following lemma.

LEMMA. *Suppose M is connected and A is a compact subset of S_1 such*

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that for each three distinct points x, y, z of M at least two of these points see a common point p of A. Then $M=S(A)$.

PROOF. Let $q\in M$. Since M is connected, there exist sequences $\{x_n\}$ and $\{y_n\}$ which converge to q and such that for each n, the points x_n, y_n and q are distinct. The hypothesis implies that, for each n, at least one of the points x_n or y_n sees a point $p_n\in A$. Since A is compact, a subsequence $\{p_n'\}$ of $\{p_n\}$ can be found which converges to $p_0\in A$ together with a corresponding subsequence (say $\{x_n'\}$) of one of the sequences $\{x_n\}$ or $\{y_n\}$ such that $x_n'p_n'\subset M$. Then every point of qp_0 is a limit point of points on the segments $x_n'p_n'$, and since M is closed, $qp_0\subset M$. Thus $M=S(A)$.

THEOREM. *Let M be a closed subset of E_r which satisfies Condition* A *with respect to a line segment S_1. Then M is the union of at most two star-shaped sets.*

PROOF. Assume that M is connected since the case where M consists of two components has already been discussed. For each positive integer k, divide S_1 into 2^k closed subintervals I_j ($j=1, 2, \cdots, 2^k$) of equal length and with at most end points in common. Then Condition A guarantees the existence of a pair of these intervals I_r and I_s such that at least two of every triple of points of M see a common point of $I_r\cup I_s$ via M. If for any k, it is possible to choose a pair I_r and I_s with $I_s=I_r$ and still preserve this property, we do so. By the lemma, $M=S(I_r)\cup S(I_s)$. Then for each k, define $I_r=I_k$ and $I_s=J_k$. It follows that for each k, $M=S(I_k)\cup S(J_k)$.

For each k, select points $x_k\in I_k$ and $y_k\in J_k$. Since S_1 is compact, we select subsequences $\{x_n\}$ of $\{x_k\}$ and $\{y_n\}$ of $\{y_k\}$ which converge to $x_0\in S_1$ and $y_0\in S_1$ respectively.

We now show $M=S(x_0)\cup S(y_0)$. Let p be an arbitrary point of M. By the lemma and the definitions of I_n and J_n, p sees a point $z_n\in I_n\cup J_n$ for each n. Without loss of generality we may assume that p sees $z_n\in I_n$ for infinitely many n. Let ϵ be any positive number and define U and V to be the respective intersections of S_1 with ϵ and $\epsilon/2$ spherical neighborhoods of x_0. Since $\{x_n\}$ converges to x_0, there exists an integer N_1 such that if $n>N_1$, $x_n\in V$. We select the integer N_2 such that if $n>N_2$, the length of I_n is less than $\epsilon/2$. Then for infinitely many $n>\max(N_1, N_2)$, $I_n\subset U$ and $pz_n\subset M$. Since p sees a point in every neighborhood of x_0 and M is closed, the segment $px_0\subset M$. Thus each point of M sees at least one of the points x_0 or y_0 via a segment in M. If for every n, $I_n=J_n$, then $x_0=y_0$ and M is star-shaped. If $x_0\neq y_0$, then M is the union of two star-shaped sets. Thus the proof is complete.

The following example shows that the assumption that M is closed cannot be deleted. Consider the union of two disjoint closed discs in the plane together with the segment joining their centers. From each disc delete all the points of a diameter not parallel to the line of centers excepting the end points and the center itself. The set described is M and S_1 is the intersection of the line of centers with M. Then M is not closed, satisfies Valentine's condition and Condition A, but it is not the union of two star-shaped sets.

Reference

1. F. A. Valentine, *Convex sets*, McGraw-Hill, New York, 1964.

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