CIRCUMSCRIBING CUBES OF A HYPERELLIPSOID[1]

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1. **Introduction.** Let C be a compact convex subset, with nonempty interior, of Euclidean n-space, R^n. To each unit vector $\mathbf{v}$, let $w(\mathbf{v})$ be the distance between the supporting hyperplanes of C normal to $\mathbf{v}$. If $\mathrm{SO}(n)$ is the group of rotations in R^n, then w induces a function $W\colon \mathrm{SO}(n)\to R^n$ as follows,

$$W(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n) = (w(\mathbf{v}_1), w(\mathbf{v}_2), \cdots, w(\mathbf{v}_n)).$$

The inverse image of the diagonal in R^n will consist of all orthogonal n-frames made up of vectors normal to the faces of a circumscribing cube about C. Each element of this set will be called *a rotation that determines a circumscribing cube around* C. In 1942, S. Kakutani [**6**] conjectured that this set was nonempty for any convex body C. This was proved by H. Yamabe and Z. Yujobo in 1950, [**10**]. Proof of Kakutani's conjecture when $n=3$ may be found in [**5**] and [**6**]; when $n=4$ in [**5**] and [**7**]. Therefore there exists a circumscribing cube around every convex body.

These results were strengthened by S. S. Cairns in 1959, [**1**] and [**2**]. The set of all circumscribing boxes around C was made into a topological space, with subspace K consisting of the circumscribing cubes. The main result in [**2**] is that the dimension of K is greater than or equal to the dimension of $\mathrm{SO}(n-1)$. Also contained there is the conjecture that K always contains a subspace homeomorphic to $\mathrm{SO}(n-1)$. This was proved when $n=3$, [**2**, p. 100]. This paper is an attempt to investigate this conjecture for $n>3$. The approach taken is to investigate the circumscribing cubes of the hyperellipsoid,

$$E = \left\{(x_1, x_2, \cdots, x_n) \,\middle|\, \sum_1^n \left(\frac{x_i}{a_i}\right)^2 \leqq 1\right\}.$$

In §2 we show that results about circumscribing cubes of centrally symmetric convex bodies are valid for arbitrary convex bodies, (Theorem 2.1). If C is centrally symmetric then every rotation that determines a circumscribing cube about C also determines a set of equal orthogonal radii of another convex body, C^*, (Theorem 2.2). So questions about circumscribing cubes are equivalent to questions about equilateral, orthogonal, inscribed frames.

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In §§3–5 only the hyperellipsoid is discussed. Theorem 3.1 states that every circumscribing cube of E has the same edge-length. This edge-length is equal to $2(1/n\sum_1^n a_i^2)^{1/2}$. Theorem 3.2 characterizes the set of unit vectors normal to faces of circumscribing cubes around E. This set of unit vectors, as a subspace of the unit $(n-1)$-sphere, is a product of spheres, $S^p \times S^q$, where $p+q=n-2$, (Theorem 4.1). Results concerning the space of rotations determining circumscribing cubes of E are found in §5, (Theorem 5.1).

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2. **General results.** The theorems of this section pertain to studying the circumscribing cubes of an arbitrary convex body, C. According to Theorem 2.1 we may assume that C is centrally symmetric. By Theorem 2.2 we can equivalently study equilateral and orthogonal radii.

Let R^n be Euclidean n-space with $\mathbf{x}\cdot y$ denoting the inner product. The *dual of* $\mathbf{x}\in R^n$ is the closed half space $\mathbf{x}^* = \{y \mid \mathbf{x}\cdot y \leqq 1\}$. If C is any convex body containing the origin then the *dual of* C is the convex body $C^* = \bigcap \mathbf{x}^*$ $(\mathbf{x}\in C)$. The *distance function of* C, $d: R^n \rightarrow R$ is defined by

$$d(\mathbf{x}) = \inf\{r > 0 \mid \mathbf{x} \in r(C)\},$$

where $r(C)$ is the convex body obtained by multiplying each vector in C by r. The *support function of* C, $s: R^n \rightarrow R$, is the distance function of C^*. If C is a closed convex body containing the origin then $C^{**} = C$ [**4**, p. 26]. Therefore the distance (support) function of C^* is the support (distance) function of C. The *width of C in the direction* $\mathbf{x}$ $(|\mathbf{x}| = 1)$ is $s(\mathbf{x}) + s(-\mathbf{x})$.

THEOREM 2.1. *To each convex set, C, there corresponds a centrally symmetric convex set, C', such that C and C' have the same width in any direction.*

Let $C' = 1/2(C-C)$, i.e. C' is the set of all vectors of the form $1/2(y-z)$ where y and z range over all points of C. C' is a centrally symmetric convex body with support function

$$S'(\mathbf{x}) = 1/2(S(\mathbf{x}) + S(-\mathbf{x})) \qquad [\mathbf{4}, \text{p. } 101].$$

Since $S'(\mathbf{x}) + S'(-\mathbf{x})$ is equal to $S(\mathbf{x}) + S(-\mathbf{x})$, the theorem follows.

THEOREM 2.2. *There is a one-to-one correspondence between the circumscribing cubes of C and sets $d_1, d_2, \cdots, d_n$ of equal and mutually orthogonal diameters of $(C')^*$.*

This is immediate from 2.1 and the fact that the support function of a convex set is the distance function of its dual.

3. **Circumscribing cubes of the hyperellipsoid.** Let E be the set of all points $(x_1, x_2, \cdots, x_n) \in R^n$ such that $\sum_1^n (x_i/a_i)^2 \leqq 1$. If the a_i are all equal then E is a solid ball and each of its circumscribing cubes has the same edge-length. The following theorem shows that this is a property shared by all hyperellipsoids.

THEOREM 3.1. *Every circumscribing cube about E has edge length equal to* $2(1/n \sum_1^n a_i^2)^{1/2}$.

We first compute the support function, s, of E. By definition, if $|\mathbf{v}| = 1$, $s(\mathbf{v}) = \mathbf{v} \cdot \mathbf{p}$ where the tangent plane of E at $\mathbf{p}$ is normal to $\mathbf{v} = (v_1, v_2, \cdots, v_n)$. Since the ith coordinate of $\mathbf{p}$ is $a_i^2 v_i/k$ where $k = (\sum_1^n (a_i e_i)^2)^{1/2}$ it follows that

$$s(\mathbf{v}) = \mathbf{v} \cdot \mathbf{p} = \frac{1}{k} \sum_1^n (a_i v_i)^2 = \left(\sum_1^n (a_i v_i)^2 \right)^{1/2}.$$

So the dual of E is the hyperellipsoid $\sum (a_i x_i)^2 \leqq 1$.

Let (x_{ij}), $i, j = 1, 2, \cdots, n$ be an orthogonal matrix, whose kth column is $\mathbf{x}_k$. The following equation implies that the sum of the squares of the edge-lengths of any circumscribing box about E is a constant.

$$(1) \qquad \sum_j s^2(\mathbf{x}_j) = \sum_j \sum_i a_i^2 x_{ij}^2 = \sum_i a_i^2 \sum_j x_{ij}^2 = \sum_i a_i^2.$$

If $\mathbf{x}_j$, $j = 1, 2, \cdots, n$ are normal to the sides of a circumscribing cube with edge-length $2e$ then

$$\sum_j s^2(\mathbf{x}_j) = n \cdot e^2 = \sum_i a_i^2.$$

Therefore $e = ((1/n) \sum_j a_i^2)^{1/2}$. From now on we reserve the symbol e for this quantity.

Theorem 3.1 states that if $\mathbf{v}$ is a unit vector normal to the face of a circumscribing cube of E then $s(\mathbf{v}) = e$. The following theorem is the converse; if $s(\mathbf{v}) = e$ then $\mathbf{v}$ is normal to the face of a circumscribing cube about E.

THEOREM 3.2. *Let $\mathbf{x}_1$ be a unit vector satisfying $s(\mathbf{x}_1) = e$; then there is an orthonormal frame $(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n)$ with $s(\mathbf{x}_i) = e$, $i = 1, 2, \cdots, n$.*

Let E' be the projection of E into the hyperplane normal to $\mathbf{x}_1$.

Let $b_1, b_2, \cdots, b_{n-1}$ be the radii of the axes of E'. From (1)

$$\sum_1^n s^2(\mathbf{x}_i) = \sum_1^{n-1} b_i^2 + e^2 = \sum_1^n a_i^2 = ne^2$$

and

$$\frac{1}{n-1}\sum_1^{n-1} b_i^2 = \frac{1}{n}\sum_1^n a_i^2.$$

Therefore circumscribing cubes of E' have the same edge length as those of E. Also, there must be a vector $\mathbf{x}_2$ normal to $\mathbf{x}_1$ such that $s(\mathbf{x}_2)=e$. Projecting E' onto the space normal to $\mathbf{x}_2$ and repeating the argument above proves the theorem. This theorem establishes the existence of a circumscribing cube about E.

Because of the duality between distance and support functions, mentioned in §2, we have the following corollary.

COROLLARY 3.3. *If $\mathbf{r}$ is a radius of E; then $\mathbf{r}$ is an element of an inscribed equilateral frame if and only if* $|\mathbf{r}| = ((1/n)\sum(1/a_i)^2)^{-1/2}$.

Let d be the distance function of E and hence the support function of E^*. Let $(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n)$ be an inscribed equilateral frame and $(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n)$ the corresponding frame of unit vectors. So $(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n)$ determines a circumscribing cube about E^* and by 3.1 $d(\mathbf{v}_i) = ((1/n)\sum(1/a_j)^2)^{1/2}$. But if $\mathbf{v}$ is a unit vector $d(\mathbf{v})$ is the reciprocal of the radius determined by $\mathbf{v}$. Therefore $|\mathbf{r}_i| = ((1/n)\sum(1/a_i)^2)^{-1/2}$, $i=1, 2, \cdots, n$.

4. **Intersection of a sphere and hyperellipsoid.** In §3 the unit vectors normal to sides of circumscribing cubes were characterized as those vectors $\mathbf{v}$ satisfying $s(\mathbf{v})=e$. In this section, the topology of this set of vectors as a subspace of the unit $(n-1)$-sphere is determined. For this consider $F: S^{n-1}\to R$ defined by

$$F(\mathbf{v}) = s^2(\mathbf{v}) = \sum_1^n (a_i v_i)^2. \tag{2}$$

The set we wish to investigate is $F^{-1}(e^2)$ which is the intersection of S^{n-1} and the hyperellipsoid $\sum_1^n (a_i x_i)^2 = e^2$. This intersection can be determined by applying Morse's critical point theory to the function F.

So that the critical points will be nondegenerate we assume that the a_i's are all distinct and that $a_i < a_{i+1}$, $i=1, 2, \cdots, n-1$. Let L_r be $F^{-1}(-\infty, r)$. The main theorem of [8, p. 2] states the following. Suppose that c is the only critical value in $[a, b]$ and that $p_1, p_2, \cdots, p_m$ are the critical points at this level with indices $k_1, k_2, \cdots, k_m$, respectively; then L_b is diffeomorphic to L_a with m handles of index k_i smoothly and disjointly attached. The function F of equation (2) has critical points $(0, \cdots, \pm 1, \cdots, 0)$ with indices $(i-1)$ where the ± 1 occurs at the ith coordinate. Therefore in dimension 3

$$\begin{aligned}
L_r &= \varnothing && \text{when } r < a_1^2,\\
L_r &= B_1 \cup B_2 && \text{when } a_1^2 < r < a_2^2,\\
L_r &= A^2 && \text{when } a_2^2 < r < a_3^2 \text{ and}\\
L_r &= S^2 && \text{when } a_3^2 < r.
\end{aligned}$$

Here B_1 and B_2 are two-dimensional disks and A^2 is a two-dimensional annulus. When $n=4$

$$\begin{aligned}
L_r &= \varnothing && \text{when } r < a_1^2,\\
L_r &= C_1 \cup C_2 && \text{when } a_1^2 < r < a_2^2,\\
L_r &= A^3 && \text{when } a_2^2 < r < a_3^2,\\
L_r &= C_1' \cup C_2' && \text{when } a_3^2 < r < a_4^2,\\
L_r &= S^3 && \text{when } a_4^2 < r,
\end{aligned}$$

where C_1, C_2, C_1', and C_2' are three dimensional disks and A^3 a three dimensional annulus. In each case the set $F^{-1}(e^2)$ is the boundary of the manifold L_r. Therefore when $n=3$, $F^{-1}(e^2)$ is a pair of 1-spheres and when $n=4$, $F^1(e^2)$ is either a pair of 2-spheres or a torus. In each case $F^{-1}(e^2)$ is a product of spheres and is dependent upon the position of e^2 among the $a_1^2, a_2^2, \cdots, a_n^2$. That this is always so, is the result of the following theorem.

THEOREM 4.1. *If $a_k < e < a_{k+1}$ then $F^{-1}(e^2)$ is homeomorphic to $S^{k-1} \times S^{n-k-1}$.*

Let S^{k-1} be the unit sphere in the k-dimensional subspace spanned by the $a_1, a_2, \cdots, a_k$ axes of E. The distance between the supporting hyperplanes normal to any vector of S^{k-1} is at most $2a_k$ and therefore is too small to determine faces of a circumscribing cube. If S^{n-k-1} is the unit sphere in the subspace spanned by the $a_{k+1}, a_{k+2}, \cdots, a_n$ axes, then these vectors determine supporting hyperplanes too far apart to be faces of a circumscribing cube.

For every $\mathbf{p} \in S^{k-1}$ and $\mathbf{q} \in S^{n-k-1}$ let $H(\mathbf{p}, \mathbf{q})$ be the two dimensional subspace spanned by the origin $\mathbf{p}$ and $\mathbf{q}$. The points $\mathbf{p}$ and $\mathbf{q}$

determine a geodesic quarter-circle $(\mathbf{p}, \mathbf{q})$ on S^{n-1}. If E^1 is the projection of E into $H(\mathbf{p}, \mathbf{q})$ then for $\mathbf{v} \in (\mathbf{p}, \mathbf{q})$, $s(\mathbf{v})$ will be the distance between the supporting lines of E^1 normal to $\mathbf{v}$. If $\mathbf{p}$ and $\mathbf{q}$ lie on the minor and major axis of E^1, then s will be strictly increasing on $(\mathbf{p}, \mathbf{q})$ and therefore must assume the value e at exactly one point, $P(\mathbf{p}, \mathbf{q})$. If the major axis of E^1 is contained in the arc $(\mathbf{p}, \mathbf{q})$, let $\mathbf{q}'$ be the reflection of q about the major axis. Since $S(\mathbf{q})$ is too large to determine faces of a cube the same must be true of every vector in the arc $(\mathbf{q}', \mathbf{q})$. However S will be strictly increasing on the arc $(\mathbf{p}, \mathbf{q}')$ and again will attain the value e at just one point, $P(\mathbf{p}, \mathbf{q})$.

Moreover the points $P(\mathbf{p}, \mathbf{q})$ are distinct for distinct pairs $(\mathbf{p}_1, \mathbf{q}_1)$ and $(\mathbf{p}_2, \mathbf{q}_2)$. Suppose $\mathbf{p}_1 \neq \mathbf{p}_2$, while $\mathbf{q}_1 = \mathbf{q}_2$; then $H(\mathbf{p}_1, \mathbf{q})$ and $H(\mathbf{p}_2, \mathbf{q})$ will intersect on the line from the origin through $\mathbf{q}$ and $P(\mathbf{p}_1, \mathbf{q}) \neq P(\mathbf{p}_2, \mathbf{q})$. The same argument applies when $\mathbf{p}_1 = \mathbf{p}_2$ and $\mathbf{q}_1 \neq \mathbf{q}_2$. Suppose now that $\mathbf{p}_1 \neq \mathbf{p}_2$ and $\mathbf{q}_1 \neq \mathbf{q}_2$. $H(\mathbf{p}_1, \mathbf{q}_1)$ and $H(\mathbf{p}_2, \mathbf{q}_2)$ intersect only at the origin, because suppose $\mathbf{t} \neq 0$ belonged to both of them. Then the line determined by the origin and $\mathbf{t}$ when projected into the subspace spanned by $a_1, a_2, \cdots, a_k$ would have to have both $\mathbf{p}_1$ and $\mathbf{p}_2$ in its image. Therefore the set of all unit vectors satisfying $s(\mathbf{v}) = e$ is homeomorphic to $S^{k-1} \times S^{n-k-1}$. This $S^{k-1} \times S^{n-k-1}$ separates S^{n-1} into a $D^k \times S^{n-k-1}$ on which s is too small and an $S^{k-1} \times D^{n-k}$ on which s is too large, [**3**, p. 25]. D^k and D^{n-k} are k and $n-k$ disks.

Because of the duality between distance and support functions and because the dual of a hyperellipsoid is a hyperellipsoid the following corollary is immediate.

COROLLARY 4.2. *The locus of the endpoints of the set of equal and orthogonal diameters of E is homeomorphic to $S^{k-1} \times S^{n-k-1}$.*

THEOREM 4.3. *The locus of the contact points of E with the faces of its circumscribing cubes is the intersection of E with the hyperellipsoid,*

$$\sum (x_i/b_i)^2 = 1/e^2 \qquad \textit{where } b_i = a_i^2, \quad i = 1, \cdots, n.$$

$(z_1, z_2, \cdots, z_n)$ will be a contact point if and only if the tangent plane, $\sum_1^n z_i x_i / a_i^2 = 1$, has distance from the origin equal to e. That is, if and only if

$$\left(\sum z_i^2/a_i^4\right)^{-1/2} = e \quad \text{or} \quad \sum z_i^2/a_i^4 = 1/e^2.$$

CONJECTURE. This locus is also a product of spheres.

5. **Rotations that determine circumscribing cubes.** The *space of circumscribing cubes* of a convex body, C, was defined in [**2**, p. 92].

An equivalent definition is the following. Let V_{nk} be the Stiefel manifold of orthogonal k-frames in Euclidean n-space, [9, p. 33]. With π_k: $V_{nk} \to V_{nk-1}$ defined by

$$\pi_k(v_1, v_2, \cdots, v_k) = (v_1, v_2, \cdots, v_{k-1})$$

we obtain the sequence

$$V_{nn} \to V_{nn-1} \to \cdots \to V_{n2} \to V_{n1}.$$

V_{nn} is the orthogonal group; V_{nn-1}, the group of rotations and V_{n1} the $(n-1)$-sphere. Let W_{nk} consist of all those k-frames of vectors normal to the faces of circumscribing cubes. We refer to W_{nn-1} as the *space of rotations that determine circumscribing cubes*. The space of circumscribing cubes is the quotient space obtained by identifying all elements of W_{nn-1} that determine the same cube, [2, p. 93].

THEOREM 5.1. *If $n > 3$ and if E is a hyperellipsoid satisfying*

$$a_1 < e < a_2 < a_3 < \cdots < a_n,$$

then W_{nn-1} is homeomorphic to 2^{n-1} disjoint copies of $V_{n-1,n-2}$.

Consider the sequence

$$W_{nn} \to W_{nn-1} \to \cdots \to W_{n2} \to W_{n1},$$

where the mappings are the restrictions of π_k to W_{nk}. By Theorem 4.1, W_{n1} is homeomorphic to $S^0 \times S^{n-2}$, the disjoint union of two $(n-2)$-spheres. According to Theorem 3.2, the orthogonal complement of each element in W_{n1} must intersect W_{n1}. Again by Theorem 4.1, this intersection must be a product of spheres, $S^p \times S^q$ where $p+q=n-3$. Since W_{n1} is not connected, this intersection is not connected and $p=0$ and $q=n-3>0$. Therefore W_{n2} is homeomorphic to four disjoint copies of $V_{n-1,2}$. Now consider the orthogonal complement of each element in W_{n2} and apply Theorem 4.1 again. The fiber over each point this time consists of a pair of $(n-4)$-spheres. Repeating this process back to W_{nn-1} proves the theorem.

COROLLARY 5.1. *If $n=5$ or 9 then W_{nn-1} is homeomorphic to 2^{n-1} copies of $S^{n-1} \times \mathrm{SO}(n-1)$.*

By [9, Corollary 8.18, p. 42], if $n=4$ or 8, $V_{nn-1} \to S^{n-1}$ is a product bundle. Since the fiber is $\mathrm{SO}(n-1)$, V_{nn-1} is homeomorphic to $S^{n-1} \times \mathrm{SO}(n-1)$.

If $n=3$, W_{32} is homeomorphic to eight copies of $V_{21}=S^1$. This is due to the fact that the fiber of $W_{32} \to W_{31}$ is $S^0 \times S^0$. W_{31} is the disjoint union of two 1-spheres and each of these 1-spheres is lifted back to W_{32} in four ways. If (v_1, v_2, v_3) is a frame of vectors all belonging to a single component, S^1, of W_{32}, then (v_3, v_1, v_2) and (v_2, v_3, v_1) are the only other rotations determining this same cube and belonging to this component. Therefore the space of circumscribing cubes is homeomorphic to the orbit space of a cyclic group of order three operating on S^1. This group is generated by a rotation through $2\pi/3$. The orbit space is again S^1. The situation in higher dimensions is not known.

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