HOLOMORPHIC APPROXIMATION ON REAL-ANALYTIC SUBMANIFOLDS OF A COMPLEX MANIFOLD

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Let X be a complex manifold, and let K be a compact subset of X. Then we will say that K is *holomorphic* if there is a sequence U_i of open Stein submanifolds of X such that $U_{i+1} \subset U_i$ and

$$K = \bigcap_{i=1}^{\infty} U_i$$

(we will use the terminology of Gunning and Rossi [2]). Let $C(K)$ be the uniform algebra of continuous complex-valued fucntions on K with respect to the maximum norm on K. Let $A_0(K)$ be the subalgebra of $C(K)$ obtained by requiring that $f \in A_0(K)$ if and only if f is the restriction to K of a holomorphic function defined in a neighborhood of K. We let $A(K)$ be the completion of $A_0(K)$ in $C(K)$. $A(K)$ is the *uniform algebra of holomorphic functions* on K. Let $S(A(K))$ be the spectrum of $A(K)$. A basic question is: when does $A(K) = C(K)$ (cf. Bishop [1])?

The purpose of this paper is to prove the following

THEOREM. *If M is a real-analytic compact submanifold of X with no complex tangent vectors, then $A(M) = C(M)$.*

This will follow immediately from the following two lemmas.

LEMMA 1. *Let M be a compact C^∞ submanifold of X with no complex tangent vectors, then M is holomorphic.*

REMARK. This theorem and the idea for its proof were suggested to me by L. Hörmander.

PROOF. Taking local coordinates $(z_1, \cdots, z_n)$ in a neighborhood of $p \in M$, with $p = (0, \cdots, 0)$, we can express M in the following manner near p. Let $z_j = x_j + iy_j$, $j = 1, \cdots, n$, and suppose $\dim_r M = k$. Set $x = (x_1, \cdots, x_k)$, $y = (y_1, \cdots, y_k)$, $Z = (z_{k+1}, \cdots, z_n)$. Then we can write M as an embedding of the form

$$y \to \begin{pmatrix} g(y) + iy \\ h(y) \end{pmatrix} \in \mathbf{C}^n,$$

where

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$$g(y) = (g_1(y), \cdots, g_k(y)),$$
$$h(y) = (h_{k+1}(y), \cdots, h_n(y))$$

are respectively real and complex-valued (vector-valued) C^∞ functions defined in a neighborhood of zero in R^k, and vanishing to second order there. Set

$$\phi = |x - g(y)|^2 + |Z - h(y)|^2.$$

We can calculate the complex Hessian of ϕ by noting that

$$x_j = (z_j + \bar{z}_j)/2$$

and

$$|h(y)| = O(|y|^2), \; |g(y)| = O(|y|^2).$$

We obtain

$$\begin{aligned} \phi_{z_j \bar{z}_j} &= \tfrac{1}{2} + O(|x| + |y|), && j = 1, \cdots, k \\ &= 1, && j = k+1, \cdots, n; \\ \phi_{z_i \bar{z}_j} &= 0, && i \neq j, \quad i \text{ or } j > k \\ &= O(|y|), && i \neq j, \quad i, j \leqq k. \end{aligned}$$

Hence it follows that for $|x| + |y|$ sufficiently small the eigenvalues of the matrix $(\phi_{z_i \bar{z}_j})$ will be positive. Thus we can find an $\epsilon > 0$ so that ϕ is strongly plurisubharmonic in $[|z| < \epsilon] = N$. We also have that ϕ vanishes in N only on M, and $d\phi = 0$ on $N \cap M$.

By compactness of M we can find a finite number of open sets $\{N_j\}$ whose union covers M, and such that there is a C^∞ function ϕ_j for each N_j, satisfying the above properties. If $\{\alpha_j\}$ is a C^∞ partition of unity subordinate to $\{N_j\}$, then it follows that $\phi = \sum \alpha_j \phi_j$ is a C^∞ function defined and strongly plurisubharmonic on a neighborhood U of M, and which vanishes in U only on M. It follows (cf. [**5**, p. 469]) that there is a sequence $\epsilon_j \to 0$ and that $U_j = [\phi < \epsilon_j]$ is a sequence of open Stein submanifolds of X whose intersection is M.

Q.E.D.

Lemma 2. *Let M be a real-analytic compact submanifold of a complex manifold X. If M is holomorphic and has no complex tangent vectors, then $A(M) = C(M)$.*[1]

Proof. Since M is holomorphic M is contained in some Stein mani-

[1] This type of result was suggested to me by H. Rossi.

fold U, open in X. Then $H(U)$, the algebra of holomorphic functions on U, separates points in U, and hence $A(K)$ separates points on K. Thus, since $A(K)$ is closed under uniform limits it will follow from the Stone-Weierstrass theorem that $A(K)=C(K)$ if we know that $f\in A(K)$ implies $\bar{f}\in A(K)$, (i.e., the algebra is self-adjoint).

Therefore it suffices to solve the following problem. Suppose F is holomorphic in a neighborhood of M, and let $f=\overline{F}|M$, we want to find a function G holomorphic in a neighborhood of M so that $G|M=f$. To construct G we have to go through several steps. Let $\dim_r M=k$ and $\dim_c X=n$.

First we note that since M has no complex tangent vectors, then $k\leqq n$. (If M is orientable, then this would follow from the holomorphicity of M, independent of the assumption about complex tangent vectors, cf. Browder [**3**].)

We now construct a "complexification" of M, that is a complex submanifold of X whose "real axis" is M.

M is defined in local coordinates near $p\in M$ as a mapping,

$$\phi: B \subset R^k \to X, \quad \phi(0) = P,$$

where ϕ is real-analytic and B is open in R^n. ϕ is then the restriction to B of a holomorphic mapping

$$\tilde{\phi}: \tilde{B} \subset \mathbf{C}^k \to X, \quad \tilde{\phi}(0) = \mathrm{p},$$

where $\tilde{B}$ is open in $\mathbf{C}^k$.

Let $R_1, \cdots, R_k$ be a basis for T_p, the real tangent space to M at p. Let J be the automorphism of τ_p, the real tangent space to X at p, which gives the complex structure to τ_p induced by the complex structure on X. Then the set of real vectors $A=\{R_1, \cdots, R_k, JR_1, \cdots, JR_k\}$ is a set of $2k$ linearly independent tangent vectors to X at p. This follows from the assumption that M has no complex tangent vectors.

We then have a holomorphic map $\tilde{\phi}$ which is nondegenerate at 0, since A is a set of $2k$ linearly independent tangent vectors to $\tilde{\phi}(\tilde{B})$ at p. Then $\tilde{\phi}(\tilde{B})$ defines in a neighborhood N of p a complex submanifold of X which contains $M\cap N$ as a submanifold. This we can do for each such set of local coordinates on M. We obtain then a k-dimensional complex submanifold V of X which contains M as a real submanifold. V is defined in U, where U is some neighborhood of M. By the holomorphicity of M, we may assume that U is an open Stein submanifold of X.

We now return to f, given above. Let N be a neighborhood of $p\in M$ in which we have local coordinates, $(t_1, \cdots, t_k)$. Since f is real-analytic in $(t_1, \cdots, t_k)$ we have that f is the restriction to $B\subset R^k$

of a holomorphic function $F_N(\zeta_1, \cdots, \zeta_k)$, defined in $\tilde{B}\subset \mathbf{C}^k$ where $(\zeta_1, \cdots, \zeta_k)$ are local coordinates for $V\cap N$. Thus we can extend f holomorphically from a neighborhood of p in M to a neighborhood of p in V, by the real-analyticity of f and M. On the intersection of two such neighborhoods the extensions to V must agree since they agree on M. Hence, we can choose a smaller neighborhood U of M (which we still require to be Stein) in which V is defined, so that we can extend f from M to a holomorphic function on V. But by the extension theorem for Stein manifolds, (see [2]) all holomorphic functions defined in V may be extended to U. Thus there exists a holomorphic function G, defined in U, with $G|M=f$. Q.E.D.

COROLLARY. *Let* Γ *be a compact real-analytic* 1-*manifold on a complex manifold* X, *then* $A(\Gamma)=C(\Gamma)$, *and hence* $S(A(\Gamma))=\Gamma$, *where* $S(A(\Gamma))$ *is the spectrum of* $A(\Gamma)$.

PROOF. A 1-manifold on X has no complex tangent vectors. The second conclusion follows from the fact that $S(C(\Gamma))=\Gamma$.

REMARK 1. If Γ is the real-analytic diffeomorphic image of S^1, the unit circle, in $\mathbf{C}^n$, then the corollary above contrasts with the results of Wermer [4] on the spectrum of the uniform algebra of polynomials on Γ, $P(\Gamma)$. In that case $S(P(\Gamma))=\Gamma$ if and only if Γ was not the boundary of a Riemann surface in $\mathbf{C}^n$ (with perhaps multiple points).

REMARK 2. It should be possible to remove the hypothesis of real-analyticity, but the method of proof would have to be entirely different. Hörmander has suggested a way of doing this by methods arising from partial differential equations.

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