SOME REMARKS ABOUT A THEOREM OF HARTOGS

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It is a theorem of Hartogs, [4, p. 231 and p. 239] and [2, Theorem 5, p. 660], that if a bounded domain $G \subset \mathbf{C}^n$, $n \geqq 2$, has a connected boundary, then any holomorphic function defined on a connected neighborhood of the boundary of G has a unique holomorphic extension to all of G. In this paper, we derive a similar theorem (Theorem 1) for a larger class of sets, the "holomorphic deficiencies" (see Definition 2) introduced by the author in [5, Definition 4.1]. Also, using a theorem of Rossi [7, Theorem 6.6, p. 464], we can show that any compact set K in a connected normal Stein space S of dimension $\geqq 2$ such that $S-K$ is connected, is a removable singularity (Theorem 3).

Other new results concern various properties of holomorphic deficiencies and envelopes of holomorphy. Perhaps the most interesting is Theorem 5: if D is an open subset of M, a Stein manifold, and $X=M-D$ is a holomorphic deficiency, then M is the envelope of holomorphy of D.

Cartan and Schwartz have shown that $H_*^1(M, \mathcal{O}) \approx 0$, where M is a Stein manifold of dimension at least 2 and $H_*^1(M, \mathcal{O})$ is the first cohomology with compact supports of M with coefficients in $\mathcal{O}$, the sheaf of germs of holomorphic functions [8, Theorem 4, p. 63]. Using this result, we shall give a direct proof of Theorem 3 for manifolds. See [8, p. 66] for another, closely related consequence of Cartan and Schwartz' result.

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Our notation is that of [3]. [6] develops the theory of envelopes of holomorphy for Riemann domains over Stein manifolds. There is a brief summary of needed theorems about such domains in [5].

DEFINITION 1. Let X and Y be point sets in M and N respectively, where M and N are complex manifolds. X and Y are *holomorphically equivalent* if they have open neighborhoods X' and Y' in M and N such that there is a biholomorphic map $r: X' \to Y'$ which maps X onto Y.

DEFINITION 2. A point set X is a complex manifold is a *holomorphic deficiency* if it is holomorphically equivalent to a set Y such that

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$Y=E-D$, where D is a Riemann domain over a Stein manifold and D is an open subset of its envelope of holomorphy E.

THEOREM 1. *Let X be a holomorphic deficiency in a complex manifold M. Let V be an open neighborhood of X. Then any holomorphic function defined on $V-X$ has a unique holomorphic extension to all of V.*

PROOF. By our definition of holomorphic deficiency, there are an open neighborhood X' of X and a biholomorphic map $r: X' \to Y'$ mapping X onto Y. $Y=E-D$, where E is the envelope of holomorphy of D. Let f be holomorphic on $V-X$. Without loss of generality, we may restrict f to $(V\cap X')-X$ and use the biholomorphic map r to consider f defined on a subset Z on the manifold E. $Z=U-Y$, where $U=r(V\cap X')$ is an open neighborhood of Y.

$E=D\cup U$. Cartan's Theorem B gives $H^1(E, \mathcal{O})\approx 0$ since E is a Stein manifold [**6**, Theorem 4.6, p. 16]. From the Mayer-Vietoris sequence [**1**, p. 236] we have the following exact sequence

$$\Gamma(E, \mathcal{O}) \to \Gamma(D, \mathcal{O}) \oplus \Gamma(U, \mathcal{O}) \to \Gamma(D \cap U, \mathcal{O}) \to 0.$$

However, $U\cap D = U\cap(E-Y) = U-Y=Z$. Thus, given a holomorphic function $f\in\Gamma(Z, \mathcal{O})$ we can write it as the difference of two holomorphic functions. $f=g_1+g_2$, $g_1\in\Gamma(D, \mathcal{O})$ and $g_2\in\Gamma(U, \mathcal{O})$. By hypothesis, g_1 has a holomorphic extension, also denoted by g_1, to all of E. We can define $f\in\Gamma(U, \mathcal{O})$ by $f=g_1-g_2$.

To prove uniqueness, consider two possible extensions f_1 and f_2. $f_1-f_2=0$ on Z and so f_1-f_2 provides a holomorphic extension of the zero function from D to all of E. This latter extension is unique since E is connected. Hence $f_1-f_2\equiv 0$ and $f_1\equiv f_2$.

We see from Theorem 1, that in order to prove that a set is a removable singularity, it suffices to show that it is a holomorphic deficiency. Let $X=E-D$ be a holomorphic deficiency with E the envelope of holomorphy of D and let $Y\subset X$ be closed in E and such that $E-Y=D\cup(X-Y)$ is connected. Then Y is a holomorphic deficiency since E is also the envelope of holomorphy of $D\cup(X-Y)$. Thus, roughly speaking, every closed subset of a removable singularity is a removable singularity.

THEOREM 2. *Let $K\subset \mathbf{C}^n$, $n\geqq 2$, be such that K is compact and $\mathbf{C}^n-K$ is connected. Let U be an open neighborhood of K. If f is a holomorphic function on $U-K$, then f has a unique holomorphic extension to U.*

PROOF. To apply Theorem 1, we must show that $\mathbf{C}^n$ is the envelope of holomorphy of $\mathbf{C}^n-K$. Enclose K in a compact polydisc Y. By [**3**, Theorem 5, p. 20], every function in $\mathbf{C}^n-Y$ extends to a holomorphic function in $\mathbf{C}^n$, which proves the assertion.

THEOREM 3. *Let K be a compact set in a normal, connected Stein space S of dimension $\geqq 2$. If $S-K$ is connected, then any holomorphic function defined on a set Z of the form $Z=U-K$, where U is an open neighborhood of K, has a unique holomorphic extension to all of U.*

PROOF. In order to adapt the proof of Theorem 1, we need only know that every holomorphic function on $S-K$ has a unique extension to all of S. This is [7, Theorem 6.6, p. 464].

We can recapture the original form of the Hartogs theorem as follows.

THEOREM 4. *Let K be a compact set in a connected, normal Stein space S of dimension $\geqq 2$. Let B, the set of noninterior points of K, be connected. Let f be holomorphic on a connected neighborhood V of B. Then there is a unique function F holomorphic in $K\cup V$ such that $F=f$ on V.*

PROOF. Let $K'=K-V$. K' is compact. $S-K'=(S-K)\cup V$ is connected. Let $U=K'\cup V=K\cup V$ and apply Theorem 3.

THEOREM 5. *If D is an open subset of M, a Stein manifold, and $X=M-D$ is a holomorphic deficiency, then M is the envelope of holomorphy of D.*

PROOF. By Theorem 1, with $V=M$, every holomorphic function on D has a unique holomorphic extension to all of M, a Stein manifold. Let E be the envelope of holomorphy of D. We must identify M with E. E is a Riemann domain over M [6]. E cannot have more than one sheet since some function f would then extend to two different values via different paths in M, which it does not. Thus we may consider $E\subset M$. Then $E=M$; for otherwise E, since it is a Stein manifold, would have holomorphic functions which do not extend to M.

THEOREM 6. *Let $D'\subset D$, with D and D' open connected sets in a Stein manifold M. Let $Y=M-D$. If $D'\cup Y$ is a connected Stein manifold, then E', the envelope of holomorphy of D', is an open subset of E, the envelope of holomorphy of D, and $E'-D'=E-D$.*

PROOF. Let $X=E-D$. X is a holomorphic deficiency. E is a Riemann domain over M. We shall identify D with its image in E. Let $\pi: E\rightarrow M$ be the projection map. Since π is a local homeomorphism $\pi(\partial X)\subset\partial Y$. Thus, since $D'\cup Y$ is a manifold, $D'\cup X$ is open in E. Theorem 1 applies to prove that every holomorphic function on D' extends uniquely to all of $D'\cup X$.

It now suffices to show that $D'\cup X$ is a Stein manifold, for then we may argue as in the proof of Theorem 4 and conclude that $E' = D' \cup X$. $\pi(X) \subset D' \cup Y$ since each holomorphic function on D' extends to X and $D' \cup Y$, being a Stein manifold, has functions which have no extension. Hence $\pi(D' \cup X) \subset D' \cup Y$. We can now show that $D' \cup X$ is holomorphically convex. Let K be a compact set in $D' \cup X$ and $\hat{K}$ the holomorphically convex hull of K with respect to the global holomorphic functions on $D' \cup X$. Since $D \cup X$ is Stein, if $\hat{K}$ is not compact it has a point in $D \cap \partial D'$ as a boundary point. But $\pi(D' \cup X) \subset D' \cup Y$ and $D' \cup Y$ is a Stein manifold and in particular holomorphically convex. Thus $\hat{K}$ cannot have a point in $D \cap \partial D'$ as a boundary point. Hence $\hat{K}$ is compact and $D' \cup X$ is holomorphically convex.

As we shall see in the concluding Corollary, the following theorem is an extension of [**3**, Corollary 3, p. 228] for Stein manifolds.

THEOREM 7. *Let X be a connected holomorphic deficiency and let B be the set of noninterior points of X. Then B is connected.*

PROOF. Let $X = E - D$ represent X as a holomorphic deficiency. Assume that B is not connected. Then $B = B_1 \cup B_2$, $B_1 \cap B_2 = \emptyset$, with $B_i \neq \emptyset$ and B_i closed in B, hence closed in X and hence closed in E. Thus, there exist disjoint open sets U_1 and U_2, with $U_1 \supset B_1$ and $U_2 \supset B_2$. Let $V_i = U_i - X$. By Theorem 1, every function f which is holomorphic on $V_1 \cup V_2$ extends uniquely to a holomorphic function on $V_1 \cup V_2 \cup X$. Let $f = 1$ on V_1 and let $f = 2$ on V_2. For $p \in B_i$, there is a polydisc neighborhood of p which is included in U_i. Hence, by the identity theorem, f must extend to be i in a neighborhood of p. So $f = i$ in a neighborhood of B_i. Now, every component of the interior of X must have a boundary which is a nonempty subset of B. By the identity theorem, f must be 1 or 2 on each component of the interior of X. We have established the existence of a holomorphic, hence continuous, function on X which assumes only the values 1 and 2. Hence X cannot be connected.

The following theorem is an easy consequence of Theorem 3 but, as mentioned above, we wish to give an independent proof. We can also deduce it from [**8**, p. 66] and Theorem 1.

THEOREM 8. *Let D be a Riemann domain over a Stein manifold of dimension $\geqq 2$ such that $\mathcal{O}_D$ separates points. Let K be a compact set in D such that $D - K$ is connected. If U is an open neighborhood of K, then every holomorphic function on $U - K$ has a unique holomorphic extension to U.*

PROOF. Since $\mathcal{O}_D$ separates points, D may be considered as an open subset of E, its envelope of holomorphy [**6**]. E is a Stein manifold and $E - K$ is connected. Thus we can replace D by E.

f is holomorphic on $U-K$. There is a C^∞ function r on E, with compact support $C\subset U$, which is identically 1 on a neighborhood V of K with $\overline{V}\subset U$. Extend rf by 0 on $E-U$. $\xi=\bar\partial(rf)$ is a C^∞ form of type (0, 1) on $E-K$ with compact support. Extend ξ to be 0 on K. Since $H_*^1(E, \mathcal{O})\approx 0$, [8, Theorem 4, p. 63], there is a (unique) C^∞ function ζ with compact support on E such that $\bar\partial\zeta=\xi$. ζ is holomorphic on V and vanishes outside some compact set C'. $\bar\partial(\zeta-rf)=0$ on $E-K$, so $\zeta-rf$ is holomorphic on $E-K$. But $\zeta-rf$ is zero outside of $C'\cup C$. Hence, since $E-K$ is connected, $\zeta\equiv rf$. But $rf=f$ on $V-K$ so that ζ provides a holomorphic extension for f to all of V.

Uniqueness is proved as before. The zero function must have a zero extension since it is the restriction to U of the zero function on $E-K$ and E is connected.

The following two results are known for Stein spaces as well as for Stein manifolds. See [3, Theorem 2, p. 227] and [3, Corollary 3, p. 228].

THEOREM 9. *Let K be a compact, holomorphically convex subset of M, a connected Stein manifold of dimension $\geqq 2$. Then $M-K$ is connected.*

PROOF. $M-K$ can have no relatively compact components. If U, a component of $M-K$, satisfied $\overline{U}$ compact, then every holomorphic function $f\in\mathcal{O}_M$ would assume its maximum modulus on $\overline{U}$ on $\partial U\subset K$. Thus U would be contained in the holomorphically convex hull of K, which is K. Thus $U=\varnothing$.

The argument in the proof of Theorem 8 now shows that every holomorphic function on $M-K$ has a unique holomorphic extension to all of M; to show that $\zeta\equiv rf$ we need just argue separately for each component. If $M-K=U_1\cup U_2$ were a decomposition of $M-K$ into disjoint open sets, then $f\equiv 1$ on V_1 and $f\equiv 2$ on V_2 would extend to a holomorphic function on M. Since M is connected, this is impossible. Thus $M-K$ has one component.

COROLLARY. *Any connected compact holomorphically convex subset K of a Stein manifold of dimension $\geqq 2$ has a connected boundary.*

PROOF. By Theorems 9 and 8, K is a holomorphic deficiency. By Theorem 7, K has a connected boundary.

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