ON AN EXAMPLE IN SECOND ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Let $b(t)$ be a given positive nondecreasing continuous function on the set $t \geqq 0$. In this note we will prove the following result:

THEOREM. *There exists a positive continuously differentiable function* $a(t)$ *such that* $a'(t) \geqq b(t)$ *and the differential equation*

$$x'' + a(t)x = 0 \qquad t \geqq 0 \qquad \left(' = \frac{d}{dt}\right), \tag{1}$$

has at least one solution $x = x(t)$ *such that*

$$\limsup_{t \to \infty} |x(t)| > 0. \tag{2}$$

The above theorem generalizes the examples given by Milloux [**4**], Hartman [**3**], and Galbraith, McShane, and Parrish [**2**], whose methods do not necessarily produce a function $a(t)$ with $a'(t) \geqq b(t)$, if $b(t)$ is taken of sufficiently large order as $t \to \infty$. Such examples are of interest in regard to the converse problem, i.e., what conditions besides $a(t) \uparrow \infty$ as $t \uparrow \infty$ need to be assumed in order to know that all solutions of (1) satisfy $x(t) \to 0$ as $t \to \infty$. The book by Cesari [**1**, pp. 84–86] has a good discussion of this problem. Willett, Wong, and Meir [**5**] list some new results in this direction. We take the occasion to point out that in [**1**, p. 86] Sansone's sufficient condition there reported should read, "If $a(t)$ is positive, nondecreasing, with a continuous derivative in $[t_0, +\infty]$, if $a'(t) \to \infty$, and $\int^{+\infty} a^{-1}(t)dt = \infty$, then for every solution $x(t)$ of (1) we have $x(t) \to 0$ as $t \to +\infty$." This corrects a misprint in [**1**, p. 86] (where "$= +\infty$" was printed as "$< \infty$").

In order to prove our main theorem, we will need the simple properties of solutions to (1) stated in the following lemma.

LEMMA. *Let* $x(t)$ *be any solution of* (1) *for a given continuous* $a(t)$, *and let* μ *and* T *be positive numbers such that* $a(t) \geqq \mu^2$ *for all* t *in* $[0, T]$. *Then* x' *has finitely many zeroes in* $[0, T]$, *and if* $t_0 < t_1 < \cdots < t_n$ *are those zeroes then* $0 < t_k - t_{k-1} \leqq 2\pi\mu^{-1}$ $(k = 1, 2, \cdots, n)$.

PROOF. By the Sturm Comparison Theorem, for any solution $x(t)$

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of (1), $x(t)$ has a finite number of zeroes in the interval $0 \leqq t \leqq T$. If τ_1 and τ_2 are successive zeroes, then $\tau_2 - \tau_1 \leqq \pi\mu^{-1}$. Now between τ_1 and τ_2, $x(t)$ is either always positive or always negative. Hence, since $x''(t) = -a(t)x(t)$, x is either strictly concave or strictly convex for $\tau_1 \leqq t \leqq \tau_2$, and so x has exactly one critical point between τ_1 and τ_2. Clearly the lemma follows.

PROOF OF THE THEOREM. Let

$$a_1(t) = 4\pi^2 + \int_0^t b(s)ds$$

for t in $[0, t_1]$, where t_1 is such that $\frac{1}{2} \leqq t_1 \leqq 1$ and $x_1'(t_1) = 0$ for $x_1(t)$ the unique solution to

$$x_1'' + a_1(t)x_1 = 0, \qquad x_1(0) = x_0 > 0, \qquad x_1'(0) = 0.$$

By the lemma, such a point t_1 must exist. For $t > t_1$ define $a_1(t) = x_1(t) = 0$. Finally, let $0 < \epsilon_n < 1$ be a given sequence of numbers such that $x_1^2(t_1) \geqq (1-\epsilon_1)x_0^2$ and $\sum_{n=1}^{\infty} \epsilon_n < \infty$.

The proof of the theorem is inductive in nature. Suppose that a set of numbers $0 = t_0 < t_1 < \cdots < t_{n-1}$ such that

$$\frac{1}{2} \leqq t_k - t_{k-1} \leqq 1 \qquad (k = 1, 2, \cdots, n-1) \tag{3}$$

and a set of functions $a_k(t)$, $x_k(t)$ $(k = 1, 2, \cdots, n-1)$ have been determined so that the following holds $(k = 1, 2, \cdots, n-1)$:

$$\begin{aligned} &x_k'' + a_k(t)x_k = 0 \quad \text{and} \quad a_k'(t) \geqq b(t) \quad \text{for} \quad t \in [t_{k-1}, t_k], \\ &x_k(t) = a_k(t) = 0 \quad \text{for} \quad t \notin [t_{k-1}, t_k], \\ &x_k(t_{k-1}) = x_{k-1}(t_{k-1}), \quad x_k'(t_{k-1}) = x_k'(t_k) = 0, \\ &a_k(t_{k-1}) = a_{k-1}(t_{k-1}), \quad a_k'(t_{k-1}) = b(t_{k-1}), \quad a_k'(t_k) = b(t_k). \end{aligned} \tag{4}$$

Suppose also that

$$x_k^2(t_k) \geqq (1 - \epsilon_k)x_k^2(t_{k-1}) \qquad (k = 1, 2, \cdots, n-1). \tag{5}$$

If we can obtain by induction a sequence of points $\{t_k\}$ and functions $\{a_k(t)\}$ and $\{x_k(t)\}$ satisfying (3), (4), and (5), the theorem will follow by taking

$$a(t) = \sum_{k=1}^{\infty} a_k(t) \quad \text{and} \quad x(t) = \sum_{k=1}^{\infty} x_k(t).$$

From (5) we obtain

$$x^2(t_k) \geqq (1 - \epsilon_k)x^2(t_{k-1}) \geqq \prod_{j=1}^{k} (1 - \epsilon_j)x_0^2 \qquad (k = 1, 2, \cdots).$$

Since $t_k \to \infty$ as $k \to \infty$ and $\sum_{j=1}^{\infty} \epsilon_j < \infty$,

$$\limsup_{t \to \infty} x^2(t) \geqq \prod_{j=1}^{\infty} (1 - \epsilon_j)x_0^2 > 0.$$

Thus, we have to show the existence of a point t_n and functions $a_n(t)$ and $x_n(t)$ such that (3), (4), and (5) hold with $k=n$. Let α be any positive number satisfying $\alpha > a_{n-1}(t_{n-1}) + b(1 + t_{n-1})$ and $\alpha > (\epsilon_n^{-1} - 1)b(1 + t_{n-1})$. For α fixed, let s_n be any number satisfying $0 < s_n - t_{n-1} < \frac{1}{2}$ and

$$s_n - t_{n-1} < \left\{ \frac{2}{\alpha} \left[1 - (1 - \epsilon_n)^{1/2}(1 + \alpha^{-1}b(1 + t_{n-1}))^{1/2}\right] \right\}^{1/2}. \tag{6}$$

Finally, let

$$a_n(t) = \int_{t_{n-1}}^{t} b(\tau)d\tau + \frac{1}{2}\left(\alpha - \int_{t_{n-1}}^{s_n} b(\tau)d\tau\right)\left(1 - \cos \pi \frac{t - t_{n-1}}{s_n - t_{n-1}}\right)$$
$$+ \frac{1}{2} a_{n-1}(t_{n-1}) \left(1 + \cos \pi \frac{t - t_{n-1}}{s_n - t_{n-1}}\right)$$

for $t_{n-1} \leqq t \leqq s_n$, and let

$$a_n(t) = \alpha + \int_{s_n}^{t} b(\tau)d\tau$$

for $s_n \leqq t \leqq t_n$. Here, t_n is any point such that $\frac{1}{2} \leqq t_n - t_{n-1} \leqq 1$ and $x_n'(t_n) = 0$ for $x_n(t)$ defined on $[t_{n-1}, t_n]$ to be the solution of

$$x_n'' + a_n(t)x_n = 0, \quad x_n(t_{n-1}) = x_{n-1}(t_{n-1}), \quad x_n'(t_{n-1}) = 0.$$

By the lemma, such a point t_n must exist. Let $x_n(t) = a_n(t) = 0$ for t not in $[t_{n-1}, t_n]$. It is easy to verify that $a_n(t)$ is a continuously differentiable function on $[t_{n-1}, t_n]$, and that $a_n(t)$ and $x_n(t)$ satisfy (4) with $k=n$.

We will now prove that $x_n(t)$ satisfies (5) with $k=n$. For the sake of brevity in what follows, let $x = x_n$ and $a = a_n$. Since $x'(t_{n-1}) = 0$, by Taylor's Theorem we obtain

$$x(s_n) - x(t_{n-1}) = \tfrac{1}{2}(s_n - t_{n-1})^2 x''(c) \quad (t_{n-1} < c < s_n).$$

Because $a' \geqq 0$, the set of maxima of $|x(t)|$ are decreasing; hence

$$|x''(c)| = a(c)|x(c)| \leqq a(s_n)|x(t_{n-1})|.$$

So

$$|x(s_n)| \geqq \left[1 - \tfrac{1}{2}(s_n - t_{n-1})^2 a(s_n)\right] |x(t_{n-1})|. \tag{7}$$

In order to estimate $|x(t_n)|$, we integrate $x'x''+axx'=0$ by parts to obtain

$$a(t_n)x^2(t_n) = [x'(s_n)]^2 + a(s_n)x^2(s_n) + \int_{s_n}^{t_n} a'(t)[x(t)]^2 dt.$$

Hence

$$x^2(t_n) \geqq \frac{a(s_n)}{a(t_n)} x^2(s_n) \geqq \frac{x^2(s_n)}{1 + \alpha^{-1}b(1 + t_{n-1})}, \tag{8}$$

since $a(s_n)=\alpha$ and

$$a(t_n) - \alpha = \int_{s_n}^{t_n} b(t)dt \leqq b(t_n)(t_n - s_n) \leqq b(1 + t_{n-1}).$$

Combining (7) and (8), we obtain

$$x^2(t_n) \geqq \frac{[1 - \frac{1}{2}(s_n - t_{n-1})^2\alpha]^2}{1 + \alpha^{-1}b(1 + t_{n-1})} x^2(t_{n-1}).$$

But from (6) it follows that

$$\frac{[1 - \frac{1}{2}(s_n - t_{n-1})^2\alpha]^2}{1 + \alpha^{-1}b(1 + t_{n-1})} > 1 - \epsilon_n.$$

Hence, $x^2(t_n) \geqq (1-\epsilon_n)x^2(t_{n-1})$, and the theorem follows.

References

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