ON THE EXTENSIONS OF CONTINUOUS FUNCTIONS FROM DENSE SUBSPACES

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Recently Taimanov and McDowell have been concerned with the question: when can a continuous function from a space X to a compact space Y be extended to a space αX in which X is dense? An important result, e.g., obtained by Taimanov is the following

THEOREM. *Let X be dense in the T_1-space αX. Then in order that a continuous function f from X into a compact space Y have a continuous extension $f^*: \alpha X \to Y$ it is necessary and sufficient that for each two disjoint closed sets F_1 and F_2, $f^{-1}[F_1]$ and $f^{-1}[F_2]$ have disjoint closures in αX.*

The following theorems provide information about the possibility of extending functions into spaces which are realcompact instead of compact.

THEOREM A. *Let αX be a T_1-space in which X is dense, and let f be a continuous function from X into a realcompact space Y. Then f has a continuous extension $f^*: \alpha X \to Y$ if and only if for every countable family $\{F_n\}$ of closed sets in Y such that $\bigcap_n F_n = \emptyset$, $\bigcap_n \mathrm{cl}_{\alpha X} f^{-1}[F_n] = \emptyset$.*

PROOF. SUFFICIENCY. We suppose some $f: X \to Y$ has no extension to all of αX. Now Y, being realcompact, is embeddable as a closed copy in R^m for some cardinal m (R denotes the reals; see [**3**, p. 160]) and we look on Y as being this subset of R^m. Consider the space $(R^m)^* = (R \cup \{\infty\})^m$ where $R \cup \{\infty\}$ is the one-point compactification of R. Then $(R^m)^*$ is compact. Then $f: X \to (R^m)^*$ also and this function does admit an extension $f^*: \alpha X \to (R^m)^*$. To show this we use Taimanov's result quoted above: Let F_1 and F_2 be disjoint closed subsets of $(R^m)^*$ and set $F_1' = F_1 \cap R^m$ and $F_2' = F_2 \cap R^m$. Then F_1, F_2 are disjoint in R^m and by hypothesis, $\mathrm{cl}_{\alpha X} f^{-1}[F_1'] \cap \mathrm{cl}_{\alpha X} f^{-1}[F_2'] = \emptyset$. But $f^{-1}[F_i] = f^{-1}[F_i']$, $i = 1, 2$ since f maps into R^m. Thus there exists a continuous extension $f^*: \alpha X \to (R^m)^*$. But then for some $p_0 \in \alpha X - X$, $f^*(p_0)$ has ∞ for at least one coordinate. Consider now the sets $F_n = (R - (-n, n))^m$, closed in the product R^m. Then $\bigcap_n F_n = \emptyset$ but $p_0 \in \bigcap_n \mathrm{cl}_{\alpha X} f^{-1}[F_n]$.

Received by the editors May 24, 1965.

[1] The authors were supported by National Science Foundation Grant No. GP-1843.

NECESSITY. Suppose $f: X \to Y$ has an extension $f^*: \alpha X \to Y$. Let $\{F_n\}$ be a family of closed sets with $\bigcap_n F_n = \emptyset$ and suppose that there is a p_0 in $\bigcap_n \mathrm{cl}_{\alpha X} f^{-1}[F_n]$. Then

$$p_0 \in \bigcap_n \mathrm{cl}_{\alpha X} f^{-1}[F_n] \subseteq \bigcap_n f^{-1}[\overline{F}_n] = \bigcap_n f^{-1}[F_n] = f^{-1}[\bigcap_n F_n] = \emptyset.$$

THEOREM B. *Let αX be a T_1-space in which X is dense and let f be a continuous function from X into a realcompact space Y. Then f has a continuous extension $f^*: \alpha X \to Y$ if and only if for any countable discrete family $\{F_n\}$ of closed sets in Y, $\{f^{-1}[F_n]\}$ is discrete in αX.*

PROOF. SUFFICIENCY. Since Y is realcompact, Y is contained as a closed copy in R^m for some cardinal m, and we consider Y to be this subset of R^m. Suppose for some $f: X \to Y$ there exists no continuous extension $f^*: \alpha X \to Y$. As in the proof of Theorem A, f, considered as a function into $(R^m)^* = (R \cup \{\infty\})^m$, has an extension $f^*: \alpha X \to (R^m)^*$. Then for some $p_0 \in \alpha X - X$, $f^*(p_0)$ has ∞ for at least one of its coordinates in $(R^m)^*$. Consider the sets $I_n = [-n-1, -n] \cup [n, n+1]$ in R and the closed sets $F_n = I_n^m$ in R^m. Then every neighbourhood of p_0 in αX must intersect infinitely many of the sets $\{f^{-1}[F_n]\}$ and also it is true that every neighbourhood of p_0 in αX must intersect infinitely many of the sets $\{f^{-1}[F_n]\}$ for n odd or for n even. Suppose the first. Then since $\{F_n: n \text{ odd}\}$ is discrete in R^m, the family $\{f^{-1}[F_n]: n \text{ odd}\}$ is discrete in αX, a contradiction.

NECESSITY. Suppose $f: X \to Y$ has an extension $f^*: \alpha X \to Y$ and consider any countable discrete family $\{F_n\}$ of closed sets in Y. Let $p_0 \in \alpha X$ and let $y_0 = f^*(p_0) \in Y$. Then y_0 has a neighbourhood U that intersects at most one F_n. Now, if $(f^*)^{-1}[U]$, which is a neighbourhood of p_0 in αX, intersected $\mathrm{cl}_{\alpha X} f^{-1}[F_1]$ and $\mathrm{cl}_{\alpha X} f^{-1}[F_2]$, say, then $f^*[(f^*)^{-1}[U]] = U$ would intersect $f^*[\mathrm{cl}_{\alpha X} f^{-1}[F_1]] = F_1$ and also $f^*[\mathrm{cl}_{\alpha X} f^{-1}[F_2]] = F_2$ since F_1 and F_2 are closed. This is a contradiction.

THEOREM C. *Let αX be a T_1-space in which X is dense, and let f be a continuous function from X into a realcompact space Y. Then f has a continuous extension $f^*: \alpha X \to Y$ if and only if*

(i) *for any two disjoint closed subsets F_1 and F_2,*

$$\mathrm{cl}_{\alpha X} f^{-1}[F_1] \cap \mathrm{cl}_{\alpha X} f^{-1}[F_2] = \emptyset;$$

(ii) *for any countable decreasing set of closed subsets $\{F_n\}$ such that*

$$\bigcap_n F_n = \emptyset, \qquad \bigcap_n \mathrm{cl}_{\alpha X} f^{-1}[F_n] = \emptyset.$$

PROOF. NECESSITY. Let $f: X \to Y$ have a continuous extension $f^*: \alpha X \to Y$.

(i) If for two closed F_1, F_2 in Y, $p_0 \in \mathrm{cl}_{\alpha X} f^{-1}[F_1] \cap \mathrm{cl}_{\alpha X} f^{-1}[F_2]$,

$$\begin{aligned} f^*(p_0) &\in f^*[\mathrm{cl}_{\alpha X} f^{-1}[F_1]] \cap f^*[\mathrm{cl}_{\alpha X} f^{-1}[F_2]] \\ &\subseteq \mathrm{cl}_Y f^*[f^{-1}[F_1]] \cap \mathrm{cl}_Y f^*[f^{-1}[F_2]] \\ &= \mathrm{cl}_Y f[f^{-1}[F_1]] \cap \mathrm{cl}_Y f[f^{-1}[F_2]] \\ &= \mathrm{cl}_Y F_1 \cap \mathrm{cl}_Y F_2 \\ &= F_1 \cap F_2. \end{aligned}$$

(ii) Let $\{F_n\}$ be decreasing, closed and such that $\bigcap_n F_n = \emptyset$. Then if $p_0 \in \bigcap_n \mathrm{cl}_{\alpha X} f^{-1}[F_n]$,

$$\begin{aligned} p_0 \in \bigcap_n \mathrm{cl}_{\alpha X}(f^*)^{-1}[F_n] &\subseteq \bigcap_n (f^*)^{-1}[\overline{F}] \\ = \bigcap_n (f^*)^{-1}[F_n] &= (f^*)^{-1}[\bigcap_n F] = \emptyset. \end{aligned}$$

SUFFICIENCY. Suppose some $f: X \to Y$ has no extension to αX. Again, $Y \subset_{\mathrm{cl}} R^m$ for some cardinal m and f considered as a function into $(R^m)^*$ (as in Theorems A and B) has an extension $f^*: \alpha X \to (R^m)^*$. Then there must be a $p_0 \in \alpha X - X$ such that $f^*(p_0)$ has at least one coordinate equal to ∞. As before, we consider the closed sets $F_n = (R - (-n, n))^m$. These are decreasing and $\bigcap_n F_n = \emptyset$ but $p_0 \in \bigcap_n \mathrm{cl}_{\alpha X} f^{-1}[F_n]$.

Notice that in the next theorem we relax the condition that the image space be realcompact.

THEOREM D. *Let X be dense in the first countable T_1-space αX and let $f: X \to Y$ continuously where Y is completely regular. Then f has a continuous extension $f^*: \alpha X \to Y$ if and only if for any two disjoint closed sets F_1 and F_2,*

$$\mathrm{cl}_{\alpha X} f^{-1}[F_1] \cap \mathrm{cl}_{\alpha X} f^{-1}[F_2] = \emptyset.$$

PROOF. SUFFICIENCY. Suppose $f: X \to Y$ has no extension. Then arguing as before, f does have an extension $f^*: \alpha X \to \beta Y$ and there must be a $p_0 \in \alpha X - X$ such that $f^*(p_0) \in \beta Y - Y$. (Here βY denotes the maximal Stone-Čech compactification of Y which exists if and only if the space Y is completely regular. See [3].)

Let $\{p_n\}$ be a sequence in X with $p_n \to p_0$. We can then select a subsequence $\{x_n\} \subset \{p_n\}$ such that the points in $\{f^*(x_n)\}$ are distinct and $f^*(x_n) \to f^*(p_0)$. Consider the closed sets

$$F_1 = \{f^*(x_1), f^*(x_3), \cdots\}$$

and

$$F_2 = \{f^*(x_2), f^*(x_4), \cdots\}.$$

Then $F_1 \cap F_2 = \varnothing$, but $\text{cl}_{\alpha X} f^{-1}[F_1] \cap \text{cl}_{\alpha X} f^{-1}[F_2] \neq \varnothing$.

NECESSITY. Let f have an extension $f^*: \alpha X \to Y$ and suppose for two closed F_1, F_2, $p_0 \in \text{cl}_{\alpha X} f^{-1}[F_1] \cap \text{cl}_{\alpha X} f^{-1}[F_2]$. We then have that

$$\begin{aligned} f^*(p_0) &\in f^*[\text{cl}_{\alpha X} f^{-1}[F_1]] \cap f^*[\text{cl}_{\alpha X} f^{-1}[F_2]] \\ &\subseteq \text{cl}_Y f^*[f^{-1}[F_1]] \cap \text{cl}_Y f^*[f^{-1}[F_2]] \\ &= \text{cl}_Y F_1 \cap \text{cl}_Y F_2 = F_1 \cap F_2, \end{aligned}$$

so their intersection is nonempty.

Added in proof. After this paper was submitted for publication, the authors have discovered that Theorem A has also been proved by R. Engelking [**4**]. On the other hand, our method can be used to prove the following generalization of Theorems A and C (the terminology is that of [**5**] and [**6**].

THEOREMS A′ & C′. *Let E be a Hausdorff space having an E-completely regular compactification E^* such that every point of $E^* - E$ has a local base of cardinality m. Let Y be E-compact, let X be a dense subspace of a T_1-space αX and let f be a continuous function with $f: X \to Y$. Then the following are equivalent*:

(a) *f admits a continuous extension $f^*: \alpha X \to Y$;*

(b) *for every class $\mathfrak{K}$ of closed subsets of Y with* card $\mathfrak{K} \leqq m$ *and $\bigcap \mathfrak{K} = \varnothing$ we have $\bigcap \{\text{cl}_{\alpha x} f^{-1}[F] : F \in \mathfrak{K}\} = \varnothing$;*

(c) *for every two disjoint closed subsets F_1 and F_2 of Y we have $\text{cl}_{\alpha x} f^{-1}[F_1] \cap \text{cl}_{\alpha x} f^{-1}[F_2] = \varnothing$ and for every submultiplicative class $\mathfrak{N}$ of closed subsets of Y with* card $\mathfrak{N} \leqq m$ *and $\bigcap \mathfrak{N} = \varnothing$ we have $\bigcap \{\text{cl}_{\alpha x} f^{-1}[F] : F \in \mathfrak{N}\} = \varnothing$.*

(A class $\mathfrak{N}$ of sets is called *submultiplicative* provided that for every F_1, $F_2 \in \mathfrak{N}$ there is an $F_3 \in \mathfrak{N}$ with $F_3 \subset F_1 \cap F_2$.)

Note that the nontrivial part of Theorem A′ and C′ [that (c) implies (a)] can be easily derived from the statement at the end of §2 in [6]. Indeed, by the first part of (c) (and by the quoted Taimanov theorem) we infer that f admits a continuous extension $f^*: \alpha X \to \beta_{E^*} Y$. Then, using the second part of (c) and the quoted statement of [**6**], we obtain that actually $f^*[\alpha X] \subset Y$.

Finally, let us call a subset F of a space Y *E-closed* provided that there exists a continuous function $f: Y \to E^n$, where n is finite, and a closed subset A of E^n with $F = f^{-1}[A]$. *Theorems* A′ & *C′ remain true if the phrase "closed subsets of Y" is replaced by "E-closed subsets of Y."*

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