A THEOREM ON THE COHOMOLOGY RING
OF NILPOTENT GROUPS

TZEE-NAN KUO[1]

Let G be a torsion free, finitely generated nilpotent group, and H be a subgroup of finite index of G. Mal'cev proved in [4] that G can be embedded in a connected, simply connected, nilpotent rational Lie group $\mathfrak{G}$, such that G and H are discrete subgroups of $\mathfrak{G}$ and the coset spaces $\mathfrak{G}/G$ and $\mathfrak{G}/H$ are compact. Nomizu showed in [5] that if L is the associated Lie algebra of $\mathfrak{G}$, then $H^n(L, Q^+) \cong H^n(\mathfrak{G}/G, Q^+)$ where Q^+ is the additive group of rational numbers. Let $H^n(G, Q^+)$ denote the rational cohomology group of the abstract group G. It follows from Lie group theory that $H^n(\mathfrak{G}/G, Q^+) \cong H^n(G, Q^+)$, hence $H^n(L, Q^+) \cong H^n(G, Q^+)$, and for the same reason we have $H^n(L, Q^+) \cong H^n(H, Q^+)$, whence $H^n(G, Q^+) \cong H^n(H, Q^+)$.

In this paper we shall, however, indicate a purely algebraic approach to this theorem in the following strengthened form:

THEOREM 1. *If G is a finitely generated nilpotent group and H is a subgroup of finite index and M is a trivial G-module which is torsion free and divisible, then $H^n(G, M) \cong H^n(H, M)$ via the restriction map for all $n \geq 0$, and if in addition M is a ring, then the restriction map is a ring isomorphism between $H^*(G, M)$ and $H^*(H, M)$.*

PROOF. Let

$$(1) \qquad \{e\} = A_0 \subset A_1 \subset \cdots \subset A_n = G$$

be a central series of G. Since the factor groups A_i/A_{i-1} of series (1) are finitely generated and abelian, it has a refinement

$$(2) \qquad \{e\} = B_0 \subset B_1 \subset \cdots \subset B_m = G$$

with cyclic factor groups B_j/B_{j-1} for all j. Let $k(G)$ be the number of infinite cyclic factor groups in (2). A finitely generated nilpotent group is an S-group in the sense of [1]. According to a result of Hirsch [1, Theorem 1.42, p. 58] $k(G)$ is independent of the choice of the central series, and if N is a normal subgroup of G, then $k(G) = k(G/N)$

$+k(N)$ in as much as the series $G \triangleright N \triangleright \{e\}$ and the series (2) have common refinement.

It is well known [3, Vol. I, p. 83] that a subgroup H of G of finite index contains a normal subgroup K of finite index in G. Since

$$H^n(G, M) \overset{\text{res}}{\cong} H^n(K, M) \qquad \text{and} \qquad H^n(H, M) \overset{\text{res}}{\cong} H^n(K, M)$$

would imply that

$$H^n(G, M) \overset{\text{res}}{\cong} H^n(H, M),$$

it suffices to show Theorem 1 under the assumption that H is normal in G.

We shall prove our theorem by induction on $k(G)$. If $k(G) = 0$, then G is a finite group. By the Universal Coefficient Theorem we have the following exact sequence

$$0 \to \text{Ext}^1(H_{n-1}(G, Z), M) \to H^n(G, M) \to \text{Hom}(H_n(G, Z), M) \to 0.$$

Since M is torsion free and divisible

$$\text{Ext}^1(H_{n-1}(G, Z), M) = \text{Hom}(H_n(G, Z), M) = 0.$$

Hence $H^n(G, M) = 0$ and similarly $H^n(H, M) = 0$ and the assertion follows. Henceforth we assume that the theorem has been proved for finitely generated nilpotent group K with $k(K) < k(G)$.

In a nilpotent group G the set A of torsion elements form a normal subgroup [3, Vol. II, p. 215]. Thus the quotient group G/A is torsion free and contains HA/A which is isomorphic to $H/H \cap A$. The group A is finite because by refining the series $G \triangleright A \triangleright \{e\}$ into a chief series

$$G \triangleright G_1 \triangleright \cdots \triangleright G_m = A \triangleright A_1 \triangleright \cdots \triangleright A_n = \{e\}$$

the factor groups A_i/A_{i+1} are necessarily finite cyclic groups. Hence $H^n(A, M) = 0$ for all $n > 0$. It follows from the Hochschild and Serre spectral sequence

$$E_2^{m,n} = H^m(G/A, H^n(A, M)) \Rightarrow H^{m+n}(G, M)$$

that $H^m(G/A, M)$ is isomorphic to $H^m(G, M)$ via the inflation homomorphism. Since $H \cap A$ is the normal subgroup of the torsion elements of H, we also have that

$$H^n(H, M) \cong H^n(H/H \cap A, M) = H^n(HA/A, M).$$

Replacing G by G/A if necessary, we may assume that G is torsion free from now on.

Let F be an infinite cyclic group contained in the intersection of the normal subgroup H and the center of the nilpotent group G. It is well known that $H^n(F, M) = 0$ if $n > 1$. Hochschild and Serre proved in [2] that

$$H^m(G/F, M) \underset{i}{\to} H^m(G, M) \underset{r'}{\to} H^{m-1}(G/F, H^1(F, M))$$

$$\underset{d_2'}{\to} H^{m+1}(G/F, M) \underset{i}{\to} H^{m+1}(G, M)$$

is an exact sequence and $d_2'(u) = -c \cdot u$ where c is the group extension

$$1 \to F \to G \to G/F \to 1$$

and i is the inflation map, while r' is induced by restricting the first argument of a suitably selected cocycle, representing the cohomology class, to H. Identifying $H^1(F, M)$ with M, we have the following commutative diagram

$$H^{m-2}(G/F, M) \to H^m(G/F, M) \to H^m(G, M)$$
$$f_1 \downarrow \qquad\qquad f_2 \downarrow \qquad\qquad f_3 \downarrow$$
$$H^{m-2}(H/F, M) \to H^m(H/F, M) \to H^m(H, M)$$
$$\to H^{m-1}(G/F, M) \to H^{m+1}(G/F, M)$$
$$f_4 \downarrow \qquad\qquad f_5 \downarrow$$
$$\to H^{m-1}(H/F, M) \to H^{m+1}(H/F, M)$$

where f_1, f_2, f_3, f_4 and f_5 are restriction maps. Since

$$k(G/F) = k(G) - k(F) = k(G) - 1 < k(G)$$

the induction hypothesis yields that f_1, f_2, f_4 and f_5 are isomorphisms. Hence by [5, lemma], f_3 is an isomorphism. If M is a ring and a and b belong to $H^*(G, M)$, then

$$\mathrm{res}(a \cdot b) = \mathrm{res}\, a \cdot \mathrm{res}\, b.$$

So the restriction map is a ring isomorphism between $H^*(G, M)$ and $H^*(H, M)$. This completes the proof.

REMARK. The integral cohomology groups of a finitely generated nilpotent group do not have the property indicated in Theorem 1 for the rational cohomology groups. Consider the finitely generated nilpotent group

$$G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in Z \right\}.$$

It contains a subgroup

$$H = \left\{ \begin{pmatrix} 1 & 2x & 2y \\ 0 & 1 & 2z \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in Z \right\}$$

of index 8. Using spectral sequence, we obtain that $H^2(G, Z) = Z \oplus Z$, whereas $H^2(H, Z) = Z_2 \oplus Z \oplus Z$. It can be easily derived from the Universal Coefficient Theorem that in general the rational cohomology ring of a finitely generated nilpotent group G equals $H^*(G, Z) \otimes Q^+$. Hence under the assumption of Theorem 1, $H^n(G, Z)$ and $H^n(H, Z)$ merely have the same number of infinite cyclic direct summands.

One might think that perhaps solvability of the group G would be sufficient to guarantee that $H^n(G, Q^+) \cong H^n(H, Q^+)$. Unfortunately, the torsion free, solvable group G generated by a and b with defining relation $aba^{-1} = b^{-1}$ has $H^1(G, Q^+)$ isomorphic to Q^+, while its subgroup H of index 2 generated by a^2 and b has $H^1(H, Q^+)$ isomorphic to $Q^+ \oplus Q^+$.

For a nilpotent group, finite generation is equivalent to the maximum condition. We would thus like to investigate the situation when the maximum condition is replaced by the minimum condition. Our result is as follows.

THEOREM 2. *If G is a solvable group with minimum condition, then $H^n(G, Q^+) = 0$ for $n > 0$.*

It follows from this theorem that under the above assumption $H^n(G, Q^+) = H^n(H, Q^+)$ for any subgroup H of G, because subgroups of G are solvable and have minimum condition.

By Cernikov's Theorem [**3**, Vol. II, p. 191], we have the exact sequence

$$1 \to H \to G \to K \to 1$$

where H is isomorphic to a finite direct sum of groups of type Z_{p^∞}, the p-primary component of the group Q^+/Z, and K is a finite solvable group. H is therefore a locally finite group. Before proving Theorem 2, we shall show the following theorem, which appears to have some interest of its own.

THEOREM 3. *If H is a locally finite group and F is a field on which H acts trivially, then*

$$H^n(H, F) = \text{proj lim } H^n(H_\alpha, F)$$

where H_α ranges over all the finitely generated subgroups of G.

PROOF. Let $Z(A)$ be the integral group ring of a given group A, and $B(A)$ denote the bar resolution of A and $\overline{B}(A)_n$ the free abelian group generated by the $Z(A)$-free generators of $B(A)_n$. It is then immediate that inj lim $Z(H_\alpha) = Z(H)$ and inj lim $\overline{B}(H_\alpha)_n = \overline{B}(H)_n$ where H_α ranges over all finitely generated subgroups of H. Since H is locally finite, H_α is necessarily finite. Hence $B(H_\alpha)_n$ is a finitely generated free abelian group and

$$\mathrm{Hom}_{Z(H_\alpha)}(B(H_\alpha)_n, F) = \mathrm{Hom}_Z(\overline{B}(H_\alpha)_n, F)$$

is a finite dimensional vector space over F. Hence

$$
\begin{aligned}
\mathrm{Hom}_{Z(H)}(B(H)_n, F) &= \mathrm{Hom}_{Z(H)}(\overline{B}(H)_n \otimes_Z Z(H), F) \\
&= \mathrm{Hom}_Z(\overline{B}(H)_n, F) \\
&= \mathrm{Hom}_Z(\mathrm{inj\,lim}\, \overline{B}(H_\alpha)_n, F) \\
&= \mathrm{proj\,lim}\, \mathrm{Hom}_Z(\overline{B}(H_\alpha)_n, F) \\
&= \mathrm{proj\,lim}\, \mathrm{Hom}_{Z(H_\alpha)}(B(H_\alpha)_n, F).
\end{aligned}
$$

Let $D(A, n)$ and $Z(A, n)$ be the subgroup of coboundaries and the subgroup of cocycles in $\mathrm{Hom}_{Z(A)}(B(A)_n, F)$ respectively. Then we have the following exact sequence of finite dimensional vector spaces:

$$0 \to Z(H_\alpha, n) \to \mathrm{Hom}_{Z(H_\alpha)}(B(H_\alpha)_n, F) \overset{\delta}{\to} \mathrm{Hom}_{Z(H_\alpha)}(B(H_\alpha)_n, F)$$
$$\to D(H_\alpha, n+1) \to 0.$$

Inverse limit preserves the exactness of a sequence of finite dimensional vector spaces, so we get the exact sequence

$$0 \to \mathrm{proj\,lim}\, Z(H_\alpha, n) \to \mathrm{Hom}_{Z(H)}(B(H)_n, F)$$
$$\overset{\delta}{\to} \mathrm{Hom}_{Z(H)}(B(H)_{n+1}, F) \to \mathrm{proj\,lim}\, D(H_\alpha, n+1) \to 0.$$

It follows that

$$\mathrm{proj\,lim}\, Z(H_\alpha, n) = Z(H, n) \quad \text{and}$$
$$\mathrm{proj\,lim}\, D(H_\alpha, n+1) = D(H, n+1).$$

Taking the inverse limit of each term in the exact sequence of finite dimensional vector spaces

$$0 \to D(H_\alpha, n) \to Z(H_\alpha, n) \to H^n(H_\alpha, F) \to 0$$

we see that

$$0 \to D(H, n) \to Z(H, n) \to \mathrm{proj\,lim}\, H^n(H_\alpha, F) \to 0$$

is exact, and the theorem follows.

COROLLARY 1. $H^n(Z_{p^\infty}, Q^+) = 0$ if $n > 0$.

PROOF. Every finitely generated subgroup of Z_{p^∞} is a cyclic group of order a power p^s of p. Since $H^n(Z_{p^s}, Q^+) = 0$ for $n > 0$, we conclude that

$$H^n(Z_{p^\infty}, Q^+) = \text{proj lim } H^n(Z_{p^s}, Q^+) = 0.$$

We now turn to the proof of Theorem 2. Since the group H is isomorphic to a finite direct sum of groups of type Z_{p^∞}, its finitely generated subgroups H_α are finite. Hence $H^n(H_\alpha, Q^+) = 0$ if $n > 0$ and

$$H^n(H, Q^+) = \text{proj lim } H^n(H_\alpha, Q^+) = 0.$$

The spectral sequence

$$E_2^{m,n} = H^m(G/H, H^n(H, Q^+)) \Rightarrow H^{m+n}(G, Q^+)$$

has $E_2^{m,n} = 0$ for $n > 0$. It follows that

$$H^m(G, Q^+) = H^m(G/H, Q^+) = H^m(K, Q^+) = 0$$

as K is a finite group. This completes the proof.

REFERENCES

1. K. Hirsch, *On infinite solvable groups*. I, Proc. London Math. Soc. (2) **44** (1938), 53–60.

2. G. Hochschild and J. Serre, *Cohomology of group extensions*, Trans. Amer. Math. Soc. **74** (1953), 110–134.

3. K. A. Kurosh, *Theory of groups*, Vols. I and II, Chelsea, New York, 1960.

4. A. I Mal'cev, *Nilpotent torsion free groups*, Izv. Akad. Nauk SSSR Ser. Mat. **13** (1949), 201–212. (Russian)

5. K. Nomizu, *On the cohomology of compact homogeneous spaces of nilpotent Lie groups*, Ann. of Math. **59** (1954), 531–538.

THE UNIVERSITY OF CHICAGO