## SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

# THE NONVANISHING OF THE BERNOULLI POLYNOMIALS IN THE CRITICAL STRIP

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Since the Bernoulli polynomials satisfy the functional equation $B_n(1-s)=(-1)^nB_n(s)$, their roots are symmetric to the critical line $\sigma=\frac{1}{2}$, where $s=\sigma+it$. The following theorem, generalizing a result of the author that $B_n(\frac{1}{2}+it)\neq 0$ for $t\neq 0$, is due to Leonard Carlitz.

THEOREM. *The Bernoulli polynomial* $B_n(\sigma+it)\neq 0$ *for* $0\leqq\sigma\leqq 1$, $t\neq 0$.

PROOF. We use the complete apparatus of Nörlund [1]. By the functional equation, it suffices to consider $0\leqq\sigma\leqq\frac{1}{2}$. For small $n$, the result is easily verified. Using Taylor's series, we have

$$
\begin{aligned}
B_n(\sigma+it) &= \sum_{k=0}^{n}\binom{n}{k}(it)^{n-k}B_k(\sigma) \\
&= \sum_{2k\leqq n}\binom{n}{2k}(it)^{n-2k}B_{2k}(\sigma) \qquad (1)\\
&\quad + \sum_{2k<n}\binom{n}{2k+1}(it)^{n-2k-1}B_{2k+1}(\sigma).
\end{aligned}
$$

Thus, if $B_n(\sigma+it)=0$, it follows that

$$
\sum_{2k<n}\binom{n}{2k+1}(it)^{n-2k-1}B_{2k+1}(\sigma) = 0. \qquad (2)
$$

But, for $0<\sigma<\frac{1}{2}$, we have $(-1)^kB_{2k+1}(\sigma)<0$, so that the signs of all the terms in (2) are the same, which shows (2) is impossible unless $t=0$. For $\sigma=0$, respectively 1, we obtain from (1)

$$
B_n(it) = -\binom{n}{2}(it)^{n-1} + \sum_{2k\leqq n}\binom{n}{2k}(it)^{n-2k}B_{2k},
$$

$$
B_n(\tfrac{1}{2}+it) = \sum_{k=0}^{n}\binom{n}{k}(2^{1-k}-1)(it)^{n-k}B_k,
$$

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and the result follows as before. A calculation gives the zero $1.272887+.0649729i$ for $B_6(s)$.


## Reference

1. N. E. Nörlund, *Mémoire sur les polynomes de Bernoulli*, Acta Math. **43** (1920), 121–131.


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# DIRECTED UNIONS AND CHAINS

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It is a well known consequence of the axiom of choice that forming unions of directed families can be reduced to (repeatedly) forming unions of chains. The most general formulation of this fact seems to be P. M. Cohn's [**1**, p. 33]: in a partially ordered set, if every well-ordered subset has a least upper bound, then every directed subset has a least upper bound. This note is mainly concerned with proving Cohn's result without the axiom of choice, but assuming the directed subset has an ordinal cardinal number. (Cohn's use of the axiom goes well beyond that.) Incidentally, well-ordered sets of the same size suffice.

It seems to be still unknown whether there exist models for set theory in which $\omega_1$ is the limit of a countable sequence of countable ordinals. In any such model, of course, when countable sequences have suprema, $\omega_1$ sequences have suprema.

THEOREM. *Let $P$ be a partially ordered set and $\omega_\alpha$ an initial ordinal such that every chain in $P$ of order type $\leqq\omega_\alpha$ has a supremum in $P$. Then every directed subset of $P$ of power at most $\aleph_\alpha$ has a supremum.*

PROOF. Let $D$ be a directed subset whose elements $x_\beta$ are indexed by ordinals $\beta<\omega_\alpha$. For each index $\beta$ and finite set $F$ of indices, we define $z(\beta, F)$ by recursion on $\beta$. Let $D(F)$ be the subset of $D$ indexed by $F$. Let $z(0, F)$ be the first-indexed $x_\gamma$ that is a common successor of the elements of $D(F)$. If $\beta=\lambda+k$ where $\lambda$ is a limit ordinal (possibly 0) and $k$ is finite nonzero, $z(\beta, F)=z(\lambda, F\cup G)$ where $G$ is the set $\{\lambda, \lambda+1, \cdots, \lambda+k-1\}$. We shall show that $z(\beta, F)$ is monotonic in

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