SEMIGROUPS AND DE LEEUW'S CONVEXITY THEOREM

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In this article the convexity theorem of de Leeuw [2, p. 194] is derived by a systematic application of the theory of Banach algebras; the "complex variables" aspect is thereby considerably reduced. The proof can be even further simplified in the case of compact Reinhardt domains in $\mathbf{C}^n$.

de Leeuw's theorem may be stated as follows: G is an abelian group with a finite set of generators $\{x_1, x_2, \cdots, x_s\}$, Γ the dual group of G, and H the semigroup generated by the set given, not necessarily including the unit of H. $P(H)$ is the semigroup algebra of H over the complex numbers. T is a compact family of homomorphisms of H into the multiplicative semigroup of complex numbers, or, as we shall say, representations of H. Define $N(h) \equiv \max\{|\phi(h)| : \phi \in T\}$, $\|\sum_{i=1}^m c_i h_i\| \equiv \sum_{i=1}^m |c_i| N(h_i)$. The norm $\|\cdot\|$ may be indefinite or incomplete, but one can consider *bounded* homomorphisms of $P(H)$ in this norm. Every complex homomorphism of $P(H)$ is given by a representation of H, schematically, $\langle \sigma, \sum_{i=1}^m c_i h_i \rangle = \sum_{i=1}^m c_i \sigma(h_i)$. The homomorphism is continuous for the given norm exactly when $|\sigma| \leqq N$. Finally, ΓT is the compact family of representations $h \to \gamma(h)\phi(h)$ with $\gamma \in \Gamma$, $\phi \in T$. Then:

THEOREM. *$\Gamma \cdot T$ contains the Šilov boundary of (the bounded homomorphisms of) $P(H)$.*

PROOF.

LEMMA 1. *Let θ, σ be representations of H, with $|\theta| \leqq 1$, and $f \in P(H)$. Then $|\langle \sigma\theta, f\rangle| \leqq \sup_{\gamma \in \Gamma} |\langle \gamma\sigma, f\rangle|$.*

PROOF. Consider $\sigma \cdot f$ and $\theta\sigma \cdot f$ as elements of the subalgebra $l_1(H) \subseteq l_1(G)$; the number on the right is just the spectral radius of $\sigma \cdot f$ in this algebra, the number on the left is at most the spectral radius of the latter. Moreover the mapping $g \to \theta \cdot g$, $g \in P(H)$, is a norm decreasing endomorphism of $P(H)$, so it does not increase the spectral radius, whence Lemma 1 follows.

COROLLARY [2, p. 195]. *If $|\theta| \leqq |\chi|$ for representations θ and χ of H, then for $f \in P(H)$, $|\langle \theta, f\rangle| \leqq \sup_{\gamma \in \Gamma} |\langle \gamma\chi, f\rangle|$.*

PROOF. Denote by S the semigroup on which $\theta \neq 0$, so that, in S,

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$\theta=\rho\chi$ for a representation ρ of S with $|\rho|\leqq 1$. Set $\rho^*(s)=\rho(s)$ for $s\in S$ $\rho^*(H\sim S)=0$. Then $\theta=\rho^*\chi$ so Lemma 1 applies.

Henceforth λ is a fixed representation of H with $|\lambda|\leqq N$; it is to be proved that for $f\in P(H)$, $|\langle\lambda, f\rangle|\leqq \sup_{\chi\in\Gamma T}|\langle\chi, f\rangle|$. Suppose S is the semigroup on which $\lambda\neq 0$; then S is generated by $S\cap\{x_1, \cdots, x_s\}$ and its characteristic function ξ_s is a representation of H. Then $\langle\lambda, f\rangle=\langle\lambda, \xi_s\cdot f\rangle$ and $\xi_s\cdot f$ belongs to the subalgebra $P(S)$; also $\sup_{\chi\in\Gamma T}|\langle\chi\xi_s, f\rangle|\leqq\sup_{\chi\in\Gamma T}|\langle\chi, f\rangle|$. All this justifies our assuming that $S=H$; this is henceforth standing hypothesis.

LEMMA 2. *Let T^+ be the subset of T of representations which vanish nowhere in H. For $h\in H$, $|\lambda(h)|\leqq\sup\{|\phi(h_0)|:\phi\in T^+\}$.*

PROOF. Write $x=x_1\cdots x_s$ and $h\in H$. For $n\geqq 1$

$$0<|\lambda(x)||\lambda(h)|^n\leqq|\phi_n(x)||\phi_n(h)|^n\leqq N(x)|\phi_n(h)|^n,$$

for some $\phi_n\in T$. Since $\phi_n(x)\neq 0$, $\phi_n\in T^+$; Lemma 2 is obtained in the limit as $n\to\infty$. In view of Lemmas 1 and 2 we may as well assume that all the representations of T are strictly positive on H; we thus abandon the assumption that T is closed.

Following [2], especially Lemmas 1.7 and 1.9, define log-co(T), the "log-convex hull" of T to be the collection of representations $\Psi>0$ with $\log\Psi\in$co$(\log T)$; that is, $\log\Psi=\sum_{i=1}^r t_i\log\phi_i$, with $\phi_i\in T$, $0<t_i\leqq 1$, and $\sum_{i=1}^r t_i=1$.

LEMMA 3. *There is a representation τ in the closure of* log-co(T) *such that* $|\lambda|\leqq\tau$.

PROOF. As in [2], define a mapping L of representations into R^s: $L(\mu)=(\log|\mu(x_1)|, \cdots, \log|\mu(x_s)|)$. Let D be the wedge of vectors with nonpositive components and E the closure of $L(T)+D$. *Claim.* $L(\lambda)\in E$. In the contrary case there is a linear function F on R^s for which $F(\xi)+\epsilon<F(L(\lambda))$ for all $\xi\in E$, $\epsilon>0$. Then F has the form $F(r_1, \cdots, r_s)\equiv\sum_{i=1}^x a_ir_i$ with each $a_i\geq 0$. Since the components of elements in E are uniformly bounded *above*, the coefficients a_i can be adjusted to be strictly positive integers. This done, $|\lambda(x_1^{a_1}\cdots x_s^{a_s})|>e^{\epsilon'}|\phi(x_1^{a_1}\cdots x_s^{a_s})|$ for every $\phi\in T$ (and $\epsilon'>0$). This contradiction proves $L(\lambda)\in E$; this means simply that there is a sequence $\{\sigma_n\}\subseteq$ log-co(T) with $\lim_{n\to\infty}|\sigma_n(x_i)|\geqq|\lambda(x_i)|$, $1\leqq i\leqq s$. τ can now be any limiting point of $\{\sigma_n\}$.

In view of the preceding lemmas it remains to prove that for τ a representation with $|\tau|\in$log-co(T), and $f\in P(H)$, $|\langle\tau, f\rangle|\leqq\sup_{\chi\in\Gamma T}|\langle\chi, f\rangle|$. The "induction step" in the proof of this is as follows. Suppose χ_1, χ_2 are positive representations and χ is a representation for which $|\chi| \leqq \max[\chi_1, \chi_2]$.

LEMMA 4. *For* $f \in P(H)$, $|\langle \chi, f\rangle| \leqq \max_{i=1,2} \sup_{\gamma \in \Gamma} |\langle \gamma\chi_i, f\rangle|$.

PROOF. To use the machinery already established, write $T=\{\chi_1, \chi_2\}$, $N=\max[\chi_1, \chi_2]$. Then χ determines a bounded homomorphism of $P(H)$ and we assume, as is clearly allowable, that χ is in the Šilov boundary of $P(H)$. The weight function N can be extended to the group G generated by H since each $\chi_i > 0$ can be so extended; consequently the norm on $P(H)$ can be extended to all of $P(G)$. The homomorphism determined by χ, being in the boundary, admits a continuous extension to $P(G)$. G being a group, it is now clear that $\log |\chi| = t \log \chi_1 + (1-t) \log \chi_2$ for some $0 \leqq t \leqq 1$. Here χ, χ_1, χ_2 denote the unique extensions of the representations to G.

Let $\{y_1, \cdots, y_u\}$ be a basis for the abelian group G, with $y_1, \cdots, y_\rho$ a basis for the torsion subgroup. For every representation ρ of G set $\zeta(\rho)=(\rho(y_1), \cdots, \rho(y_u))$ so that ζ is a homeomorphism into $\boldsymbol{C}^u$ of the representations of G. The range $\mathfrak{M}$ of the transformation ζ, applied to the representations whose moduli are dominated by N, is determined as follows: $\zeta_1, \cdots, \zeta_\rho$ are restricted to certain finite subgroups while $(\log |\zeta_{\rho+1}|, \cdots, \log |\zeta_u|)$ is restricted to a certain compact convex subset A of $R^{u-\rho}$. With $(\zeta_1, \cdots, \zeta_\rho)$ held fixed, each $f \in P(H)$ determines in an obvious way an analytic function F of $(\zeta_{\rho+1}, \cdots, \zeta_u)$ on an open set containing these coordinates of $\mathfrak{M}$. Recalling that A is the convex hull of $\{(\log \chi_i(y_{p+1}), \cdots, \log \chi_i(y_u)) : i=1, 2\}$, Lemma 4 follows from the fact that $\log F$ is subharmonic along any complex plane. For the details of the "three lines theorem" in n variables, cf. Dunford and Schwartz [1, p. 521]. This completes Lemma 4, which may be compared with de Leeuw's Lemma 1.9. q.e.d.

ADDENDUM. In the case that H is the additive semigroup of s-tuples of positive integers, one observes that it is sufficient to consider compact sets T whose coordinates are bounded away from zero, that is, $\{\phi^{-1} : \phi \in T\}$ is compact. Extend T to the group of lattice points and proceed as in Lemma 4.

REFERENCES

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