MAXIMAL IDEAL SPACES AND A-CONVEXITY

D. R. WILKEN[1]

1. **Introduction.** Let X be a compact Hausdorff space. $C(X)$ denotes the Banach algebra of all complex valued continuous functions on X with the supremum norm. A subalgebra A contained in $C(X)$ is called a *function algebra on* X if it satisfies the following conditions.

(i) A separates points on X.

(ii) The constant functions are in A, i.e., 1 belongs to A.

(iii) A is closed in $C(X)$.

For a function algebra A the space of maximal ideals is denoted by M_A and the Šilov boundary by S_A. Both S_A and M_A carry natural compact Hausdorff topologies and S_A and X can be imbedded in M_A so that, identifying S_A and X with their images in M_A, we have $S_A \subset X \subset M_A$. Moreover A, via restriction, is a function algebra on S_A and, via the Gelfand representation, extends to a function algebra on M_A. S_A and M_A are respectively the "smallest" and "largest" compact Hausdorff spaces on which A can be realized as a function algebra.

If A and B are function algebras on M_A with A a subalgebra of B, frequently it is of interest to know when the maximal ideal space of B coincides with that of A. For example, this is the case when one wishes to apply the local maximum modulus principle to B. Thus, when considering a function algebra B as above, it would be helpful to have criteria for determining the relation of M_B to M_A. In this direction Stolzenberg [**4**] has shown that if the functions in B agree locally with functions in A, then M_B coincides with M_A. The main purpose of this note is to give a method of determining M_B explicitly in terms of M_A in many cases, and to present a direct way of constructing function algebras B as above with larger maximal ideal spaces. Included is a counterexample to a question apparently raised by Hoffman and referred to by Glicksberg in [**1**].

2. A-**convexity.** For an arbitrary function algebra A with maximal ideal space M_A we use A_F to denote the uniform closure on a closed set F in M_A of the restriction algebra of A to F.

2.1. Definition. For a closed set F in M_A, the *A-convex hull of F in M_A*, denoted M_F, is

Received by the editors January 6, 1966.

[1] This research was supported in part by NSF Grant GE-6587 and in part by a NSF Postdoctoral Fellowship.

$$M_F = \{x \in M_A: |g(x)| \leqq \|g\|_F, \text{ for all } g \in A\}$$

($\|g\|_F = \sup_F |g|$). F is said to be *A-convex* if $M_F = F$.

Thus M_F is the maximal ideal space of the function algebra A_F (cf. [3]) and F is A-convex if and only if $M_{A_F} = F$. For a finitely generated algebra A, the concept of A-convexity coincides with that of polynomial convexity in the space of n-complex variables.

2.2. THEOREM. *Let A be a function algebra on M_A, let f be in $C(M_A)$ and let B be the function algebra generated by A and f on M_A. Let $R(f)$ denote the range of f on M_A. For z in $R(f)$, let $T_z = \{x \in M_A : f(x) = z\}$. Then either T_z is A-convex for every z in $R(f)$ or M_B properly contains M_A. (Since B is a function algebra on M_A, M_A is imbedded in M_B. It is in terms of this imbedding that the final statement of the theorem is to be interpreted.)*

PROOF. Let M_z denote the A-convex hull of T_z in M_A. Let $K \subset M_A \times \mathbf{C}$ be defined by $K = \{(x, f(x)) : x \in M_A\}$. ($\mathbf{C}$ denotes the complex plane.) Then K is an imbedding of M_A in $M_A \times \mathbf{C}$. Let

$$H = \{(x, w) \in M_A \times \mathbf{C}: x \in M_w\}.$$

We claim that H can be imbedded in M_B. Let r denote the canonical retraction of M_B onto M_A, i.e., $r: M_B \to M_A$ satisfies $x = r(y)$ if and only if $g(x) = g(y)$ for all g in A. (Since $A \subset B$, the functions in A extend continuously to functions on M_B.) Define a mapping $s: M_B \to M_A \times \mathbf{C}$ by $s(m) = (r(m), f(m))$. We show that s is an imbedding of M_B into $M_A \times \mathbf{C}$ such that $H \subset s(M_B)$.

(i) s is one-to-one. If $s(m_1) = s(m_2)$, then $r(m_1) = r(m_2)$ so that $g(m_1) = g(r(m_1)) = g(r(m_2)) = g(m_2)$ for all g in A. Also $f(m_1) = f(m_2)$. Hence $h(m_1) = h(m_2)$ for all h in B and $m_1 = m_2$.

(ii) $H \subset s(M_B)$. For $(x, w) \in H$ we define a multiplicative functional $m(x, w)$ on B as follows. Let B' be the dense subalgebra of B of all polynomials in f with coefficients from A. Define $m(x, w)$ on B' by

$$m(x, w)\left(\sum a_i f^i\right) = \sum a_i(x) w^i, \qquad a_i \in A.$$

Then $m(x, w)$ is clearly a multiplicative linear functional on B'. To extend it to B it suffices to show $m(x, w)$ is bounded on B'. But

$$\left| m(x, w)\left(\sum a_i f^i\right) \right| = \left| \sum a_i(x) w^i \right| \leqq \left\| \sum a_i w^i \right\|_{T_w} = \left\| \sum a_i f^i \right\|_{T_w}$$

since $\sum a_i w^i \in A$ and $(x, w) \in H$, i.e., $x \in M_w$. Thus $m(x, w)$ extends to a multiplicative functional on B. Also for g in A, $m(x, w)(g) = g(x)$ implies $r(m(x, w)) = x$; and $m(x, w)(f) = w$. Therefore $s(m(x, w)) = (x, w)$ and $H \subset s(M_B)$.

(iii) $s(M_A)=K$. For $m\in M_A\subset M_B$, $r(m)=m$. Hence $s(m)=(m,f(m))\in K$. If we now identify M_A and M_B with their imbeddings under s in $M_A\times\mathbf{C}$ we have $M_A\subset H\subset M_B$. To obtain the theorem we need now merely observe that if T_z is not A-convex for some z, then M_z properly contains T_z so that H, and hence M_B, properly contains M_A.

Evidently the theorem gives a direct method of exhibiting function algebras A and B satisfying $A\subset B\subset C(M_A)$ and with M_B properly larger than M_A. We can also show that in many cases the set H described in the proof of the theorem actually fills out M_B so that we have an explicit description of M_B in terms of M_A. To this end we prove the following lemma.

2.3. LEMMA. *Let A, f, and B be as in Theorem 2.2. If $f(M_B)$ properly contains $f(M_A)$, then each point w in $f(M_B)-f(M_A)$ lies in a bounded component of $\mathbf{C}-f(M_A)$. Moreover $f(M_B)$ is the union of $f(M_A)$ with those bounded components of $\mathbf{C}-f(M_A)$ which meet $f(M_B)$.*

(In other words $f(M_A)$ can only be enlarged to $f(M_B)$ by completely filling in some holes.)

PROOF. Let $m\in M_B$ be such that $f(m)$ is not in $f(M_A)$. If $f(m)$ lies in the unbounded component of $\mathbf{C}-f(M_A)$, then there is a polynomial p in one complex variable such that $|p(f(m))|>\|p\|_{f(M_A)}$. Since $g=p(f)$ is an element of B we have $|g(m)|>\|g\|_{M_A}$. This contradicts the fact that B as a function algebra on M_A has its Šilov boundary contained in M_A. To finish the lemma suppose $f(m)$ lies in some bounded component U of $\mathbf{C}-f(M_A)$. If U is not contained in $f(M_B)$ there is a point w on the boundary of $f(M_B)$ and contained in $U-f(M_A)$. Let $w=f(m_0)$. Choose a in $U-f(M_B)$ such that

$$|a-w|<\inf_{m'\in M_A}|w-f(m')|.$$

Then $g(z)=1/(z-a)$ is analytic on a neighborhood of $f(M_B)$ so that $g(f)\in B$. But $|g(f(m_0))|>\|g(f)\|_{M_A}$, which, as above, is impossible.

2.4. THEOREM. *Let H be as described in the proof of Theorem 2.2, i.e., $H=\{(x,w)\in M_A\times\mathbf{C}: x\in M_w\}$. If $f(M_A)$ is a compact, nonseparating subset of the plane without interior, then $M_B=H$.*

PROOF. By Lemma 2.3, $f(M_B)=f(M_A)$. As in Theorem 2.2 consider the imbedding $s: M_B\to M_A\times\mathbf{C}$ given by $s(m)=(r(m),f(m))$ where r is the canonical retraction of M_B onto M_A. Then

$$s(M_A)\subset H\subset s(M_B)\subset M_A\times f(M_A)$$

and we wish to show $H=s(M_B)$. If $s(m)\in s(M_B)-H$, from the definition of H, there exists a function g in A such that $1=|g(r(m))|$

$>\|g\|_{M_{f(m)}}$. Since $s(m) \notin s(M_A)$ we can choose a neighborhood V of $f(m)$ in $\mathbf{C}$ such that

$$(t, \beta) \in s(M_A) \cap [M_A \times (V \cap f(M_A))] \Rightarrow |g(t)| < |g(r(m))|.$$

Also since $f(M_A)$ is a compact nonseparating subset of $\mathbf{C}$ without interior every continuous function on $f(M_A)$ can be uniformly approximated on $f(M_A)$ by polynomials. In particular if h is a function which peaks at $f(m)$, i.e.,

$$h(f(m)) = 1, \ |h(z)| < 1 \quad \text{for} \quad z \in f(M_A), z = f(m),$$

then h can be approximated by polynomials. But $f(M_A) = f(M_B)$ implies $h(f)$ is in B. Choose an integer N such that $\|h^N\|_{f(M_A)-V} < 1/\|g\|$. Let $h' = h^N(f)g$. Then $h' \in B$, $|h'(m)| = 1$ and $\|h'\|_{M_A} < 1$, a contradiction as in Lemma 2.3.

2.5. Corollary. *If f is real-valued on M_A, then $M_B = H$.*

Each of the above statements has an immediate extension to a more general situation in which a finite number of functions are adjoined to A. Then the statements of the results have to be formulated in terms of intersections of the level sets of the functions. As an indication we formulate Corollary 2.5 in the more general setting. Let $f_1, f_2, \cdots, f_n \in C(M_A)$, let

$$T(w_1, \cdots, w_n) = \{x \in M_A : f_i(x) = w_i, i = 1, 2, \cdots, n\},$$
$$M(w_1, \cdots, w_n) = \text{the } A\text{-convex hull of } T(w_1, \cdots, w_n)$$

and

$$H = \{(x, w_1, \cdots, w_n) \in M_A \times C^n : x \in M(w_1, \cdots, w_n)\}.$$

2.5′. Corollary. *If $f_1, f_2, \cdots, f_n$ are real-valued on M_A, and B is the function algebra generated by A and $f_1, f_2, \cdots, f_n$ on M_A, then $M_B = H$.*

Corollary 2.5 contains as a special case an observation made by Mergelyan [2] concerning the disc algebra. He observed that the set of maximal ideals of the algebra generated by the polynomials and a real function on the unit disc in the complex plane coincides with the disc just in the case that the level sets of the real function do not divide the plane.

3. **Examples.** The results above give an explicit way of relating the maximal ideal spaces of certain algebras and one naturally inquires whether, in the situations considered, it is possible to deduce

that this relation always obtains. That is, in the terminology of §2, is it always the case that $M_B = H$? In this section we describe some simple examples which show that Theorem 2.4 cannot be essentially strengthened. We also include the counterexample mentioned in the introduction.

3.1. EXAMPLE. Let $D = \{z \in \mathbf{C}: |z| \leqq 1\}$, $\Gamma = \{z: |z| = 1\}$ and let A be the disc algebra consisting of the uniform limits of polynomials on D. Let $f \in C(D)$ be given by $f(z) = \exp(2\pi i\, |z|)$, and let B be the function algebra generated by A and f on D. That is, if f_0 is the identity function on D, $f_0(z) = z$, then B is the algebra generated by f and f_0. We have $A \subset B \subset C(D)$, $M_A = D$, and $f(M_A) = \Gamma$ is a compact subset of $\mathbf{C}$ without interior. For $w \in f(M_A)$, $w = \exp(2\pi i\, t)$,

$$T_w = \{z \in D: f(z) = w\} = \{z: |z| = t\}.$$

Hence $M_w = \{z \in D: |z| \leqq t\}$, the A-convex hull of T_w and

$$H = \{(z, w) \in D \times \mathbf{C}: z \in M_w\} = \{(z, \exp(2\pi i\, t)): 0 \leqq |z| \leqq t \leqq 1\}.$$

Let $s: M_A \to \mathbf{C}^2$ be defined by $s(z) = (z, \exp(2\pi i\, |z|))$ and consider the following subsets of $D \times D \subset \mathbf{C} \times \mathbf{C} = \mathbf{C}^2$:

$$X = s(D) = \{z, \exp(2\pi i\, |z|)): |z| \leqq 1\},$$

$$Y = \text{the polynomially convex hull of } X \text{ in } \mathbf{C}^2.$$

Then Y coincides with the maximal ideal space of B and $X \subset H \subset Y$. We assert that Y is properly larger than H. To see this it suffices to show $(0, 0) \in Y$. But if $p(w_1, w_2)$ is any polynomial on $\mathbf{C}^2$, then

$$|p(0, 0)| \leqq \sup_{|w_2| = 1} |p(0, w_2)| \leqq \sup_X |p(w_1, w_2)|.$$

It is not difficult to see that, in fact, $Y = H \cup E$ where $E = \{(0, w_2): |w_2| \leqq 1\}$ and that the rationally convex hull of X in $\mathbf{C}^2$ is H. Thus this simple example also exhibits the interesting phenomenon of a disc in $\mathbf{C}^2$ whose rationally convex hull lies properly between itself and its polynomially convex hull. Geometrically H looks like a solid cone bent around so that its tip touches the center of its base.

To give an example such that $f(M_A)$ has interior (but does not separate the plane) in which M_B differs from H, take A as the polynomial algebra on two disjoint discs in the plane, and adjoin the function which is the identity on the first disc and maps the second disc onto the boundary of the first.

Our final example is a counterexample to the following question (see [1]):

3.2. QUESTION. Let $A \subset B \subset C(M_A)$ where A and B are function algebras with the same Šilov boundary. Is $M_B = M_A$?

We give a negative reply which is a modification of a special case of Corollary 2.5 according to the observation that for any function algebra A on M_A it is possible to construct a function algebra A' satisfying:

(i) $M_{A'} = S_{A'}$, i.e., the Šilov boundary and the maximal ideal space of A' coincide.

(ii) There is an imbedding of M_A in $M_{A'}$ such that $A' | M_A = A$. To obtain this let I be the unit interval and consider M_A as identified with the subset $M_A \times \{0\}$ of $M_A \times I$. Extend A to A' on $M_A \times I$ by setting

$$A' = \{f \in C(M_A \times I) : f | (M_A \times \{0\}) \in A\}.$$

Direct verification shows $M_{A'} = M_A \times I = S_{A'}$.

3.3. EXAMPLE. Let A and D be as in Example 3.1. Let $f(z) = |z|$ and let B be the function algebra generated by A and f on D. Then by Corollary 2.5, M_B is the cone

$$M_B = \{(z, t) : 0 \leqq |z| \leqq t \leqq 1\} \subset \mathbf{C}^2,$$

so that M_B properly contains M_A. Let A' be constructed from A as above, i.e.,

$$A' = \{f \in C(D \times I) : f | (D \times \{0\}) \in A\}$$

and let

$$B' = \{f \in C(D \times I) : f | (D \times \{0\}) \in B\}.$$

Then $M_{A'} = M_A \times I = S_{A'} = S_{B'}$, $A' \subset B' \subset C(M_{A'})$, but $M_{B'}$ is the cylinder with the bottom slice replaced by the cone which we have identified with M_B. Thus $M_{B'}$ properly contains $M_{A'}$.

REFERENCES

1. I. Glicksberg, *Maximal algebras and a theorem of Radó*, Pacific J. Math. **14** (1964), 919–941.

2. S. N. Mergelyan, *Uniform approximations to functions of a complex variable*, Uspehi Mat. Nauk **7** (1952), no. 2 (48), 31–122; English transl., Amer. Math. Soc. Transl. **101** (1954), 19.

3. H. Rossi, *The local maximum modulus principle*, Ann. of Math. **72** (1960), 1–11.

4. G. Stolzenberg, *The maximal ideal space of the functions locally in a function algebra*, Proc. Amer. Math. Soc. **14** (1963), 342–345.

TULANE UNIVERSITY AND
BRANDEIS UNIVERSITY