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A SELECTION THEOREM

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1. **Introduction.** The following theorem was proved in [**1**, Footnote 7]. (A function ϕ from X to the collection 2^B of nonempty closed subsets of B is called *lower semicontinuous* (=*l.s.c.*) if $\{x\in X: \phi(x)\cap V\neq\emptyset\}$ is open in X whenever V is open in B, while $\Gamma_B A$ denotes the closed convex hull of A in B.)

THEOREM 1.1 [**1**]. *If X is paracompact, if B is a Banach space, and if ϕ: $X\to 2^B$ is l.s.c., then there is a continuous f: $X\to Y$ such that $f(x)\in\Gamma_B\phi(x)$ for every $x\in X$.*

As was pointed out in [**1**, p. 364], Theorem 1.1 remains true if B is any complete, metrizable locally convex space, but it is generally false if B is not metrizable. We can, however, prove the following generalization of Theorem 1.1.

THEOREM 1.2. *Let X be paracompact, and M a metrizable subset of a complete*[2] *locally convex space E. Let ϕ: $X\to 2^M$ be l.s.c. and such that, for some metric on M, every $\phi(x)$ is complete. Then there exists a continuous f: $X\to E$ such that $f(x)\in\Gamma_E\phi(x)$ for every $x\in X$.*

Theorem 1.2 was proved in [**3**] under the stronger assumption that X is metrizable. While that was sufficient for the applications in [**3**], and probably for most other applications, it did not generalize Theorem 1.1, and was therefore never entirely satisfying. In this

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[2] It suffices if $\Gamma_E K$ is compact for every compact $K\subset M$.

paper, some machinery is created in §2 which enables us to prove Theorem 1.2 in full generality.

2. **Two lemmas.** Let M be a metric space with metric ρ, and L the linear space of real-valued Lipschitz functions on X. As was shown in [**4**], there is a Banach space B containing M isometrically, and a linear map $f \to f^*$ from L to the dual space of B, such that $f^* | M = f$ for all $f \in L$.

For each $f \in L$, let $s(f) = \{x \in M : f(x) \neq 0\}^-$. For each $y \in B$, let

$$\mathcal{S}(y) = \{U \subset M : U \text{ open}, f^*(y) = 0 \text{ whenever } f \in L \text{ and } s(f) \subset U\},$$

and let $\sigma(y) = M - \bigcup \mathcal{S}(y)$. Clearly $\sigma(y)$ is closed.

The following lemmas are not the sharpest results possible, but they suffice for our purposes.

LEMMA 2.1. *Suppose that $K \subset M$ is compact, and that $y \in \Gamma_B K$. Then*
(a) $\sigma(y) \subset K$.
(b) *If $f \in L$, and $\sigma(y) \cap s(f) = \varnothing$, then $f^*(y) = 0$.*
(c) $\sigma(y) \neq \varnothing$.

PROOF. (a) Let $U = M - K$. Then $f^*(y) = 0$ whenever $s(f) \subset U$, so $U \in \mathcal{S}(y)$ and hence $\sigma(y) \subset K$.

(b) Since $f = f^+ - f^-$, where $f^+ \geqq 0$ and $f^- \geqq 0$ and both are in L, we need only prove (b) for $f \geqq 0$.

Let $A = s(f) \cap K$. Then A is compact and disjoint from $\sigma(y)$, so it can be covered by open $V_i \subset X$ $(i = 1, \cdots, n)$ such that $\overline{V}_i \subset U_i$ for some $U_i \in \mathcal{S}(y)$. Let $g_i(x) = \rho(x, X - V_i)$ for all $x \in M$. Then $g_i \in L$ and $s(g_i) \subset U_i$, so $g_i^*(y) = 0$. Let $g = \sum_{i=1}^n g_i$. Then $g^*(y) = 0$. Now $g(x) > 0$ when $x \in A$, so there is an $\alpha > 0$ such that $0 \leqq f(x) \leqq \alpha g(x)$ for all $x \in A$, and therefore for all $x \in K$ since $f(x) = 0$ if $x \in K - A$. Hence $0 \leqq f^*(y) \leqq \alpha g^*(y) = 0$.

(c) Let $h(x) = 1$ for all $x \in M$. Then $h \in L$ and $h^*(y) = 1$, so $\sigma(y) \cap s(h) \neq \varnothing$ by (b). Hence $\sigma(y) \neq \varnothing$, and that completes the proof.

Now let $H = \bigcup \{\Gamma_B K : K \subset M, K \text{ compact}\}$, let $\mathcal{K}(M)$ be the set of compact elements of 2^M, and let $\sigma : H \to \mathcal{K}(M)$ be the map $y \to \sigma(y)$.

LEMMA 2.2. *The map $\sigma : H \to \mathcal{K}(M)$ is l.s.c.*

PROOF. Let $V \subset M$ be open, and suppose that $y_0 \in H$ and $\sigma(y_0) \cap V \neq \phi$. Then there is an $f_0 \in L$ such that $s(f_0) \subset V$ and $f_0^*(y_0) \neq 0$. Let $U = \{y \in H : f_0^*(y) \neq 0\}$. Then U is a neighborhood y_0 in H, and if $y \in U$, then $\sigma(y) \cap s(f_0) \neq \varnothing$ by Lemma 2.1 (b), so $\sigma(y) \cap V \neq \varnothing$. Hence $\{y \in H : \sigma(y) \cap V \neq \varnothing\}$ is open in H, so σ is l.s.c.

3. **Proof of Theorem 1.2.** First, apply [**2**, Theorem 1.1] to pick a

l.s.c. map $\psi: X \to 2^M$ such that, for all $x \in X$, we have $\psi(x) \subset \emptyset(x)$ and $\psi(x)$ is compact.

Let $B \supset H \supset M$ be as in §2, and apply Theorem 1.1 to pick a continuous $g: X \to B$ such that $g(x) \in \Gamma_B \psi(x)$ for all $x \in X$. Then $g(x) \in H$ for all $x \in X$.

Let $\sigma: H \to \mathcal{K}(M)$ be as in §2. Apply [**3**, Theorem 1.2] (that is, our Theorem 1.2 with *metrizable* domain) to the set-valued function σ, which is l.s.c. by Lemma 2.2, to pick a continuous $h: H \to E$ such that $h(y) \in \Gamma_E \sigma(y)$ for all $y \in H$.

Define $f: X \to E$ by $f = goh$. If $x \in X$, then $g(x) \in \Gamma_E \psi(x)$, so $\sigma(g(x)) \subset \psi(x) \subset \phi(x)$ by Lemma 2.1 (a), and hence

$$f(x) = h(g(x)) \in \Gamma_E \sigma(g(x)) \subset \Gamma_E \phi(x).$$

That completes the proof.

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