ON THE BOUNDEDNESS OF SOLUTIONS OF SECOND ORDER NONLINEAR DIFFERENTIAL SYSTEMS

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1. **Introduction.** The purpose of this paper is to obtain sufficient conditions for the boundedness of solutions of equations of the form

$$x'' + B(t)x' + A(t)x^{2k-1} = 0, \tag{1.1}$$

where x is a column m-vector,

$$x = (x_1, x_2, \cdots, x_m),$$
$$x^{2k-1} = (x_1^{2k-1}, x_2^{2k-1}, \cdots, x_m^{2k-1}), \qquad \text{for } k \text{ a positive integer.}$$

$A(t)$, $B(t)$ are continuous m by m matrices with elements which are continuous real-valued functions of t for t on I: $a \leqq t < +\infty$. Since m can be replaced by $2m$ there is no loss of generality in assuming that the m^2-elements of $A(t)$, $B(t)$ are real valued. Thus the components of the vectors on which $A(t)$, $B(t)$ operate will be confined to the real field.

For a fixed t on I: $a \leqq t < +\infty$ let $2H(t)$ be the sum and $2K(t)$ the difference of $A(t)$ and its transposed matrix $A^T(t)$, that is, $A(t) = H(t) + K(t)$, where $H(t)$ is symmetric and $K(t)$ is skew-symmetric. If $A(x, y)$ is the bilinear form belonging to the matrix $A(t)$ the form $K(x, x)$ vanishes identically. Let $\circ$ denote scalar multiplication of vectors, then

$$\begin{aligned} f(t)\,|x|^2 \leqq x \circ Ax &= A(x, x) = H(x, x) + K(x, x) \\ &= x \circ Ax \leqq g(t)\,|x|^2 \end{aligned} \tag{1.2}$$

where $g(t)$ is the greatest eigenvalue of $H(t)$, $f(t)$ is the smallest eigenvalue of $H(t)$, and $|x|^2$ is the square of the length of x. If t varies over I, then $H(t)$, $f(t)$, $g(t)$ are continuous functions of t since $A(t)$ is a continuous function of t. $x \circ y = \sum_{i=1}^m x_i y_i$, for $x = (x_1, x_2, \cdots, x_m)$, $y = (y_1, y_2, \cdots, y_m)$. Only nonidentically zero solutions of (1.1) will be considered. Also the notation $(x \circ x) = |x|^2$ for the square of the length of a vector will be used.

For the case $k = 1$, P. Hartman [2] considered the question of the existence of large or small solutions of equations of the type (1.1) and A. Wintner [6] established a comparison theorem for Sturmian oscillation numbers of solutions of equations of the type (1.1).

Received by the editors May 23, 1966

J. Jones [**3**] obtained a sufficient condition for nonoscillatory solutions of (1.1) for the case of $B(t) \equiv 0$, and $k=1$. Equations of the type (1.1) arise in the investigation of the natural frequencies, rates of decay and modes of free vibrations of damped systems. For a recent treatment for the case of constant matrices for $k=1$ see P. Lancaster [**4**].

2. **Results.** We have the following results concerning the boundedness of solutions $x(t)$ of (1.1) on the interval $I = a \leqq t < +\infty$.

THEOREM 1. *Let the following conditions hold:*

(i) *$B(t)$ is a continuous m by m matrix with real-valued elements on I,*

(ii) *$A(t)$ is continuously differentiable, symmetric, positive definite and nondecreasing on I,*

(iii) *$A'(t)+B(t)A(t)+A(t)B(t)^T \geqq 0$ on I, where T denotes the transposed matrix, and prime $'$ denotes differentiation with respect to t.*

Then $E[x(t)] = (x^k \circ x^k) + k(y \circ y)$ is nonincreasing on I.

$A(t)$ is nondecreasing on I if $t \geqq s$, and $A(t) - A(s)$ is nonnegative. Let $A^{1/2}(t)$ be the unique symmetric positive definite square root of $A(t)$, and $A^{-1/2}(t) = (A^{1/2}(t))^{-1}$. Now $A(t)$ differentiable and $A(t) > 0$ on I imply that $A^{1/2}(t)$ is also differentiable on I.

Equation (1.1) may be written as a system of first order differential equations for a $(2m)$-vector (x, y) where

$$y = A^{-1/2}(t)x'. \tag{2.1}$$

An equivalent system to (1.1) is the following

$$\begin{aligned} x' = A^{1/2}(t)y,\ y' = &- A^{1/2}(t)x^{2k-1} - A^{-1/2}(t)(A^{1/2}(t))'y \\ &- A^{-1/2}(t)B(t)A^{1/2}(t)y. \end{aligned} \tag{2.2}$$

A solution vector $x = x(t)$ of (1.1) determines a solution vector $(x, y) = (x(t), y(t))$ of (2.2) and conversely. For any solution (x, y) of (2.2), let

$$\begin{aligned} E(t) \equiv E[x(t)] &\equiv (x^k \circ x^k) + k(y \circ y) \\ &= (x^k \circ x^k) + k(A^{-1}(t)x' \circ x') \end{aligned} \tag{2.3}$$

be an amplitude functional. Making use of (1.1), (2.3) and the identity

$$A^{-1/2}(t)[A^{1/2}(t)A^{1/2}(t)]'A^{-1/2}(t) = A^{-1/2}(t)A'(t)A^{-1/2}(t),$$

we have

$$
\begin{aligned}
\frac{dE}{dt} &= -2k[A^{1/2}(t)(A^{1/2}(t))' + A^{-1/2}(t)B(t)A^{1/2}(t)]y \circ y \\
&= -k[A^{-1/2}(t)(A^{1/2}(t))' + (A^{1/2}(t))'A^{-1/2}(t) \\
&\qquad + 2A^{-1/2}(t)B(t)A^{1/2}(t)]y \circ y \\
&= -kA^{-1/2}(t)[A'(t) + B(t)A(t) + A(t)B(t)^T]A^{-1/2}(t)y \circ y.
\end{aligned} \tag{2.4}
$$

Now by (iii), (1.1), (2.4) we see that $E(t)$ is nonincreasing on I.

THEOREM 2. *Let conditions* (i), (ii) *of Theorem* 1 *hold along with the following conditions*

$$
\text{(iv)} \quad \int_a^\infty [f(t)]^- dt < +\infty, \quad [f(t)]^- = \frac{|f(t)| - f(t)}{2}, \quad [f(t)]^+ = \frac{|f(t)| + f(t)}{2}
$$

for $t \in I$ where $f(t)$ is the smallest eigenvalue of the symmetric component of

$$
A^{-1/2}(t)[A'(t) + B(t)A(t) + A(t)B^T(t)]A^{-1/2}(t), \tag{2.5}
$$

then all solutions $x(t)$ of (1.1) *remain bounded as $t \to +\infty$.*

Now

$$
\begin{aligned}
0 \leqq (x^k \circ x^k) + k(y \circ y) &\equiv E[x(t)] = E[x(a)] \\
- k\int_a^t \{A^{-1/2}(\tau)[A'(\tau) &+ B(\tau)A(\tau) + A(\tau)B^T(\tau)]A^{-1/2}(\tau)\} y \circ y d\tau.
\end{aligned} \tag{2.6}
$$

Using (1.2), (2.6) we have

$$
0 \leqq k|y|^2 \leqq E[x(a)] + k\int_a^t [f(\tau)]^- \cdot |y|^2 d\tau, \tag{2.7}
$$

where $f(t)$ is the smallest eigenvalue of the symmetric component of (2.5), and $|y|^2$ is the square of the length of the vector y. Using an inequality given in R. Bellman [1, p. 35, (2.7)] we have

$$
\begin{aligned}
0 \leqq k|y|^2 &\leqq E[x(a)] \exp\left(\int_a^t [f(\tau)]^- d\tau\right) \\
&\leqq E[x(a)] \exp\left(\int_a^\infty [f(t)]^- dt\right).
\end{aligned} \tag{2.8}
$$

Thus $|y|$ is bounded as $t \to +\infty$. Then from (2.6) we have

$$0 \leqq (x^k \circ x^k) \leqq (x^k \circ x^k) + k(y \circ y) \equiv E[x(t)]$$

$$\leqq E[x(a)] + k\int_a^t [f(\tau)]^- \cdot |y|^2 d\tau - k\int_a^t [f(\tau)]^+ \cdot |y|^2 d\tau, \tag{2.9}$$

$$k\int_a^\infty [f(\tau)]^+ \cdot |y|^2 d\tau \leqq E[x(a)] + k\int_a^\infty [f(t)]^- \cdot |y|^2 dt,$$

and

$$E[x(t)] \leqq E[x(a)] + k\int_a^t [f(\tau)]^- \cdot |y|^2 d\tau - k\int_a^t [f(\tau)]^+ \cdot |y|^2 d\tau, \tag{2.10}$$

hence $E[x(t)]$ remains bounded as $t \to +\infty$ and thus all solutions remain bounded as $t \to +\infty$.

THEOREM 3. *Let A, B be m by m constant matrices having real elements, A positive definite and symmetric, and C any m by m positive definite matrix having real elements, such that the following pairs of $2m$ by $2m$ matrices are not similar, namely,*

$$\begin{pmatrix} B & C \\ 0 & -B^T \end{pmatrix}, \quad \begin{pmatrix} B & 0 \\ 0 & -B^T \end{pmatrix} \tag{2.11}$$

then $E[x(t)] = (x^k \circ x^k) + k(y \circ y)$ is nondecreasing on I for $x(t)$ a solution of (1.1).

W. E. Roth [5] has shown that the similarity of the pair of matrices of (2.11) is a necessary and sufficient condition that the matrix equation

$$BX + XB^T = C \tag{2.12}$$

have a solution X. Thus no solution X of (2.12) exists for any positive definite matrix C and (2.3), (2.4) imply that $E(t)$ is a nonnegative nondecreasing function of t for $t \in I$.

REFERENCES

1. R. Bellman, *Stability theory of differential equations*, McGraw-Hill, New York, 1953; p. 35.

2. P. Hartman, *The existence of large or small solutions of linear differential equations*, Duke Math. J. **37** (1961), 421–430.

3. John Jones, Jr., *On oscillation numbers of second order linear differential systems*, Quart. Appl. Math. **23** (1965), 181–183.

4. P. Lancaster, *Bounds for latent roots in damped vibration problems*, SIAM Rev. (2) **6** (1964), 121–126.

5. W. E. Roth, *The equations $AX-YB=C$, and $AX-XB=C$ in matrices*, Proc. Amer. Math. Soc. **3** (1952), 392–396.

6. A. Wintner, *A comparison theorem for Sturmian oscillation numbers of linear systems of second order*, Duke Math. J. **24** (1958), 515–519.

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ORDER ISOMORPHISMS OF CONES

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Let C be a closed convex cone of vertex ϕ in a real normed vector space L. We suppose C does not contain an entire straight line, and let $\cdot \geqq \cdot$ be the order induced on L by C. Let S be some subset of L. A map $\varphi: S \rightarrow S$ is said to be *order preserving* on S if $x \geqq y$ implies $\varphi(x) \geqq \varphi(y)$. We say a bijection $\varphi: S \leftrightarrow S$ is *regular* on S if both φ and φ^{-1} are order preserving on S. (We are not assuming φ continuous.)

Zeeman has shown [2] that if C is a right circular cone in $\boldsymbol{R}^4$, the only regular maps of $\boldsymbol{R}^4$ are affine, with linear part a Lorentz transformation. In this paper we show that similar conclusions can be drawn under more general circumstances.

Before proceeding, we would like to record the benefit of several useful conversations with M. Koecher. We also note that, at the suggestion of the referee, we have modified our Proposition 1 to include the infinite dimensional case.

As part of the setting for the sequel, we shall insist that C have a *compact base*. This means there is a continuous linear functional h on L such that $h(x)>0$ for $x \in C-\phi$, and the set $P=h^{-1}(1) \cap C$, called a *base* of the cone, is compact. Each ray in C intersects P exactly once, and as is customary, we call a ray passing through an extreme (respectively exposed) point of P an extreme (respectively exposed) ray of C.

Since P is the closure of the convex hull of its extreme points, C is the closure of the convex hull of its extreme rays. Furthermore, let p be an exposed point of P. There is, by definition, a continuous linear functional s on L and a real number α such that $s(x)+\alpha \geqq 0$ for $x \in P$,

Received by the editors February 9, 1966.