ON A THEOREM OF KLEE

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In 1953 and 1956, Klee [3], [4] proved that for $E$ any infinite-dimensional normed linear space and $K$ any compact subset of $E$, $E \setminus K$ is homeomorphic to $E$. Klee’s argument used sequences of bounded convex sets. In [5], Lin has some extensions of Klee’s results using modifications of his methods. In this paper we give a short and elementary proof of a somewhat more general result\(^1\) using only simple set-theoretic properties.

A space $S$ is said to be an $\alpha$-space provided

1. $S$ is an infinite-dimensional topological linear space, i.e., an infinite dimensional real vector space with a Hausdorff topology in which vector addition and scalar multiplication are jointly continuous,

2. $S$ has a Schauder basis, i.e., a sequence $\{x_i\}_{i>0}$ of elements of $S$ such that for each $s \in S$ there is a unique sequence of scalars $\{a_i\}$ with $s = \sum_{i=1}^{\infty} a_i x_i$ (convergence being in the topology of $S$) such that the function $f_i$ defined by $f_i(s) = a_i$ is continuous for each $i$, and

3. there is a neighborhood $U$ of the origin such that the elements $\{x_i\}$ of the Schauder basis above are not in $U$.

Henceforth, all spaces under discussion are to be $\alpha$-spaces.

For each $i$, let $M_i$ denote the product of $i$ copies of the reals with usual distance function $d_i$ referring to distance between points, between a point and a set or between two sets. Let $f_i$ be as defined in condition (2) of the definition of an $\alpha$-space and let $g_j$ be the map of $S$ onto $M_j$ defined by $g_j(s) = (f_1(s), f_2(s), \cdots, f_i(s))$. Since, by hypothesis, $f_i$ is continuous (for each $i$), then so is $g_j$ for each $j$.

A set $K \subseteq S$ is said to be projectible provided

1. $K$ is closed,

2. for any $p \notin S \setminus K$, there is a $j$ such that $g_j(p)$ is not an element of the closure of $g_j(K)$, and

3. there exist infinitely many $i$ such that $f_i(K)$ is bounded above or below.

The proof of the following lemma is trivial and is therefore omitted.

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Lemma. Let $S$ be an $\alpha$-space, $s \in S$, $K$ be a projectible subset of $S$, and let $g_i(s)$ be a point of $M_i$ not in the closure of $g_i(K)$. Then, for each $j > i$, $d_j(g_i(s), g_i(K)) \geq d_i(g_i(s), g_i(K)) > 0$.

Theorem. Let $S$ be an $\alpha$-space and $K$ be a projectible subset of $S$. Then $S \setminus K$ is homeomorphic to $S$.

Proof. Without loss of generality, we may let $\{n_i\}_{i>0}$ be a sequence of integers such that, for each $i > 0$, $n_i > i$ and $f_{n_i}(K)$ is bounded below by $\varepsilon_{n_i}$. For each $i > 0$, let $U_i$ be the $(1/2^i)$-neighborhood of $g_i(K)$ in $M_i$.

For each $i \geq 1$, let $h_i$ be the homeomorphism of $S$ onto itself such that (1) for each $q \in S$ and each $m \neq n_i$, $f_m(h_i(q)) = f_m(q)$ and (2) for each $q \in S$, $f_{n_i}(h_i(q)) = f_{n_i}(q) + 2r_{q,i}(\varepsilon_{n_i} + i)$ where

$$r_{q,i} = \min \left[ \frac{d_i(g_i(q), M_i \setminus U_i)}{d_i(g_i(K), M_i \setminus U_i)}, 1 \right].$$

Let $h$ be defined as follows:

for $j \in \{n_i\}$, $f_j(h(s)) = f_j(s)$

for any $i > 0$, $f_{n_i}(h(s)) = f_{n_i}(h_i(s))$.

Then $h$ is a homeomorphism of $S \setminus K$ onto $S$ as desired and as we shall verify.

Consider $q \in S \setminus K$. By condition 2 of projectibility and the Lemma, there is a neighborhood $V$ of $q$ such that $g_i(V) \subset M_i \setminus U_i$ for all but finitely many $i$'s. Thus, for all but finitely many $i$'s, $h_i$ is the identity on $V$. Hence $h(q)$ is an element of $S$ and $h$ is continuous at $q$.

Let $\pi_i$ be a homeomorphism defined coordinatewise as follows:

for $j \leq i$, $f_{n_j}(\pi_i(x)) = f_{n_j}(h_j(x))$

for $k \in \{n_j\}_{j=1}^i$, $f_k(\pi_i(x)) = f_k(x)$.

We note that for $j > 0$, $h_j$ is the identity except on $g_j^{-1}(U_j)$. Also $g_j^{-1}(U_j) \supset g_j^{-1}(U_{j+1})$. Thus $(\pi_{j+1} \pi_j^{-1})$ is the identity except on $\pi_j(g_j^{-1}(U_{j+1}))$ since $\pi_{j+1}$ acts in the same way as $\pi_j$ except on $\pi_j(g_j^{-1}(U_{j+1}))$.

Clearly by considering successive coordinates, $h$ may be regarded as $-(\pi_{j+1} \pi_j^{-1})-(\pi_2 \pi_1^{-1})(\pi_1)\pi_1$ and for each $i$, $\pi_i$ is the product of the first $i$ indicated factors from the right. Now we think of the effects of these factors starting from the right. Let $iU$ denote the set of products of the scalar $i$ and the elements of the set $U$ of condition (3) of the $\alpha$-space definition.
First $\pi_1$ moves $g_2^{-1}(U_2)$ outside of $1U$ in the $n_1$ direction. Thus $(\pi_2\pi_1^{-1})$ is the identity on $1U$. But $(\pi_2\pi_1^{-1})$ moves $\pi_1(g_2^{-1}(U_3))$ outside of $2U$ in the $n_2$ direction. Thus $(\pi_3\pi_2^{-1})$ is the identity on $2U$. Inductively, $(\pi_i\pi_{i-1}^{-1})$ is the identity on $iU$. But since, for each $i$ and each $j>0$, $(\pi_{i+j}\pi_{i+j-1}^{-1})$ is the identity on $iU$, then on $iU$, $h^{-1}$ may be considered to be defined as

$$[(\pi_{i+1}\pi_{i}^{-1})(\cdots(\pi_2\pi_1^{-1})\pi_1)]^{-1}.$$ 

Hence since $S=\bigcup_{i>0} iU$, $h^{-1}$ is defined and continuous on $S$ and $h$ is a homeomorphism of $S\setminus K$ onto $S$.

It is clear that any Banach space with a basis is an $\alpha$-space (for, without loss of generality, we may assume that the basis elements all have norm 1). Thus for each $p\geq 1$, $l_p$ is an $\alpha$-space. It is not hard to see that all $l_p$ spaces for $0<p<1$ are also $\alpha$-spaces. In [6], it is shown that various topological linear spaces including some nonmetrizable ones satisfy conditions guaranteeing that they are $\alpha$-spaces. On the other hand, the countable infinite product $s$ of lines as a topological linear space is not an $\alpha$-space (the set $U$ does not exist). The argument of this paper does not work for this type of space. However, in [1] the author shows by a different argument that any countable union of compact sets or even sets comparable to projectible sets may be deleted from $s$ without changing its topological character. Since $l_2$ is homeomorphic to $s$, [2], $l_2$ also can lose an arbitrary countable union of compact sets without changing its topological character. Indeed, Klee's argument [3] can be easily modified to show that for $K$ any countable set of points, $l_2\setminus K$ is homeomorphic to $l_2$.

For any $\alpha$-space $S$, any compact set $K$ is projectible since, if, for each $i>0$, $g_i(q)$ is an element of the closure of $g_i(K)$, then as $K$ is compact, $q$ is a limit point of $K$. Clearly there are many projectible sets which are not compact. In all Banach spaces with bases, all weakly (sequentially) compact sets are projectible.

**Corollary.** If $S$ is an $\alpha$-space and $K$ is a compact subset of $S$, then $S\setminus K$ is homeomorphic to $S$.

**Corollary.** If $S$ is a Banach space with a basis and $K$ is a weakly compact subset of $S$, then $S/K$ is homeomorphic to $S$.

**References**

A SELECTION THEOREM

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1. Introduction. The following theorem was proved in [1, Footnote 7]. (A function φ from X to the collection 2^B of nonempty closed subsets of B is called lower semicontinuous (=l.s.c.) if \{x \in X: φ(x) \cap V \neq \emptyset\} is open in X whenever V is open in B, while Γ_B A denotes the closed convex hull of A in B.)

**Theorem 1.1** [1]. If X is paracompact, if B is a Banach space, and if φ: X → 2^B is l.s.c., then there is a continuous f: X → Y such that f(x) ∈ Γ_B φ(x) for every x ∈ X.

As was pointed out in [1, p. 364], Theorem 1.1 remains true if B is any complete, metrizable locally convex space, but it is generally false if B is not metrizable. We can, however, prove the following generalization of Theorem 1.1.

**Theorem 1.2.** Let X be paracompact, and M a metrizable subset of a complete locally convex space E. Let φ: X → 2^M be l.s.c. and such that, for some metric on M, every φ(x) is complete. Then there exists a continuous f: X → E such that f(x) ∈ Γ_E φ(x) for every x ∈ X.

Theorem 1.2 was proved in [3] under the stronger assumption that X is metrizable. While that was sufficient for the applications in [3], and probably for most other applications, it did not generalize Theorem 1.1, and was therefore never entirely satisfying. In this case, it suffices if Γ_E K is compact for every compact K ⊆ M.

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