ON A THEOREM OF KLEE

R. D. ANDERSON

In 1953 and 1956, Klee [3], [4] proved that for E any infinite-dimensional normed linear space and K any compact subset of E, $E\backslash K$ is homeomorphic to E. Klee's argument used sequences of bounded convex sets. In [5], Lin has some extensions of Klee's results using modifications of his methods. In this paper we give a short and elementary proof of a somewhat more general result[1] using only simple set-theoretic properties.

A space S is said to be an α-space provided

(1) S is an infinite-dimensional topological linear space, i.e., an infinite dimensional real vector space with a Hausdorff topology in which vector addition and scalar multiplication are jointly continuous,

(2) S has a Schauder basis, i.e., a sequence $\{x_i\}_{i>0}$ of elements of S such that for each $s \in S$ there is a unique sequence of scalars $\{a_i\}$ with $s=\sum_{i=1}^{\infty} a_i x_i$ (convergence being in the topology of S) such that the function f_i defined by $f_i(s)=a_i$ is continuous for each i, and

(3) there is a neighborhood U of the origin such that the elements $\{x_i\}$ of the Schauder basis above are not in U.

Henceforth, all spaces under discussion are to be α-spaces.

For each i, let M_i denote the product of i copies of the reals with usual distance function d_i referring to distance between points, between a point and a set or between two sets. Let f_i be as defined in condition (2) of the definition of an α-space and let g_i be the map of S onto M_i defined by $g_i(s)=(f_1(s), f_2(s), \cdots, f_i(s))$. Since, by hypothesis, f_i is continuous (for each i), then so is g_j for each j.

A set $K \subset S$ is said to be *projectible* provided

(1) K is closed,

(2) for any $p \in S\backslash K$, there is a j such that $g_j(p)$ is not an element of the closure of $g_j(K)$, and

(3) there exist infinitely many i such that $f_i(K)$ is bounded above or below.

The proof of the following lemma is trivial and is therefore omitted.

Presented to the Society, November 21, 1964; received by the editors March 20, 1965.

[1] I am indebted to V. L. Klee and to J. R. Retherford for helpful suggestions leading to the present form of the main theorem (which was originally conceived as a theorem about l_p spaces).

LEMMA. *Let S be an α-space, $s\in S$, K be a projectible subset of S, and let $g_i(s)$ be a point of M_i not in the closure of $g_i(K)$. Then, for each $j>i$, $d_j(g_j(s), g_j(K))\geqq d_i(g_i(s), g_i(K))>0$.*

THEOREM. *Let S be an α-space and K be a projectible subset of S. Then $S\backslash K$ is homeomorphic to S.*

PROOF. Without loss of generality, we may let $\{n_i\}_{i>0}$ be a sequence of integers such that, for each $i>0$, $n_i>i$ and $f_{n_i}(K)$ is bounded below by z_{n_i}. For each $i>0$, let U_i be the $(1/2^i)$-neighborhood of $g_i(K)$ in M_i.

For each $i\geqq 1$, let h_i be the homeomorphism of S onto itself such that (1) for each $q\in S$ and each $m\neq n_i$, $f_m(h_i(q))=f_m(q)$ and (2) for each $q\in S$, $f_{n_i}(h_i(q))=f_{n_i}(q)+2r_{q,i}(|z_{n_i}|+i)$ where

$$r_{q,i} = \min\left[\frac{d_i(g_i(q),\ M_i\backslash U_i)}{d_i(g_i(K),\ M_i\backslash U_i)},\ 1\right].$$

Let h be defined as follows:

$$\text{for } j \notin \{n_i\},\ f_j(h(s)) = f_j(s)$$
$$\text{for any } i > 0,\ f_{n_i}(h(s)) = f_{n_i}(h_i(s)).$$

Then h is a homeomorphism of $S\backslash K$ onto S as desired and as we shall verify.

Consider $q\in S\backslash K$. By condition 2 of projectibility and the Lemma, there is a neighborhood V of q such that $g_i(V)\subset M_i\backslash U_i$ for all but finitely many i's. Thus, for all but finitely many i's, h_i is the identity on V. Hence $h(q)$ is an element of S and h is continuous at q.

Let π_i be a homeomorphism defined coordinatewise as follows:

$$\text{for } j \leqq i, \qquad f_{n_j}(\pi_i(x)) = f_{n_j}(h_j(x))$$
$$\text{for } k \notin \{n_j\}_{j=1}^{i}, \qquad f_k(\pi_i(x)) = f_k(x).$$

We note that for $j>0$, h_j is the identity except on $g_j^{-1}(U_j)$. Also $g_j^{-1}(U_j)\supset g_{j+1}^{-1}(U_{j+1})$. Thus $(\pi_{j+1}\pi_j^{-1})$ is the identity except on $\pi_j(g_{j+1}^{-1}(U_{j+1}))$ since π_{j+1} acts in the same way as π_j except on $\pi_j(g_{j+1}^{-1}(U_{j+1}))$.

Clearly by considering successive coordinates, h may be regarded as—$(\pi_{j+1}\pi_j^{-1})$—$(\pi_3\pi_2^{-1})(\pi_2\pi_1^{-1})\pi_1$ and for each i, π_i is the product of the first i indicated factors from the right. Now we think of the effects of these factors starting from the right. Let iU denote the set of products of the scalar i and the elements of the set U of condition (3) of the α-space definition.

First π_1 moves $g_2^{-1}(U_2)$ outside of $1U$ in the n_1 direction. Thus $(\pi_2\pi_1^{-1})$ is the identity on $1U$. But $(\pi_2\pi_1^{-1})$ moves $\pi_1(g_3^{-1}(U_3))$ outside of $2U$ in the n_2 direction. Thus $(\pi_3\pi_2^{-1})$ is the identity on $2U$. Inductively, $(\pi_{i+1}\pi_i^{-1})$ is the identity on iU. But since, for each i and each $j>0$, $(\pi_{i+j}\pi_{i+j-1}^{-1})$ is the identity on iU, then on iU, h^{-1} may be considered to be defined as

$$[(\pi_i\pi_{i-1}^{-1})(\cdots(\pi_2\pi_1^{-1})\pi_1)]^{-1}.$$

Hence since $S=\bigcup_{i>0} iU$, h^{-1} is defined and continuous on S and h is a homeomorphism of $S\backslash K$ onto S.

It is clear that any Banach space with a basis is an α-space (for, without loss of generality, we may assume that the basis elements all have norm 1). Thus for each $p\geq 1$, l_p is an α-space. It is not hard to see that all l_p spaces for $0<p<1$ are also α-spaces. In [**6**], it is shown that various topological linear spaces including some nonmetrizable ones satisfy conditions guaranteeing that they are α-spaces. On the other hand, the countable infinite product s of lines as a topological linear space is not an α-space (the set U does not exist). The argument of this paper does not work for this type of space. However, in [**1**] the author shows by a different argument that any countable union of compact sets or even sets comparable to projectible sets may be deleted from s without changing its topological character. Since l_2 is homeomorphic to s, [**2**], l_2 also can lose an arbitrary countable union of compact sets without changing its topological character. Indeed, Klee's argument [**3**] can be easily modified to show that for K any countable set of points, $l_2\backslash K$ is homeomorphic to l_2.

For any α-space S, any compact set K is projectible since, if, for each $i>0$, $g_i(q)$ is an element of the closure of $g_i(K)$, then as K is compact, q is a limit point of K. Clearly there are many projectible sets which are not compact. In all Banach spaces with bases, all weakly (sequentially) compact sets are projectible.

COROLLARY. *If S is an α-space and K is a compact subset of S, then $S\backslash K$ is homeomorphic to S.*

COROLLARY. *If S is a Banach space with a basis and K is a weakly compact subset of S, then S/K is homeomorphic to S.*

REFERENCES

1. R. D. Anderson, *Topological properties of the Hilbert cube and the infinite product of open intervals*, Trans. Amer. Math. Soc. (to appear).

2. ———, *Hilbert space is homeomorphic to the countable infinite product of lines*, Bull. Amer. Math. Soc. **72** (1966), 515–519.

3. V. L. Klee, *Convex bodies and periodic homeomorphism in Hilbert spaces*, Trans. Amer. Math. Soc. **74** (1953), 10–43.

4. ———, *A note on topological properties of normed linear spaces*, Proc. Amer. Math. Soc. **7** (1956), 673–674.

5. Bor-Luh Lin, *Two topological problems concerning infinite-dimensional normed linear spaces*, Trans. Amer. Math. Soc. **114** (1965), 156–175.

6. J. R. Retherford and C. W. McArthur, *Some remarks on bases in linear topological spaces*, Math. Ann. **64** (1966), 38–41.

LOUISIANA STATE UNIVERSITY

A SELECTION THEOREM

E. MICHAEL[1]

1. **Introduction.** The following theorem was proved in [**1**, Footnote 7]. (A function ϕ from X to the collection 2^B of nonempty closed subsets of B is called *lower semicontinuous* (=*l.s.c.*) if $\{x \in X: \phi(x) \cap V \neq \emptyset\}$ is open in X whenever V is open in B, while $\Gamma_B A$ denotes the closed convex hull of A in B.)

THEOREM 1.1 [**1**]. *If X is paracompact, if B is a Banach space, and if ϕ: $X \to 2^B$ is l.s.c., then there is a continuous f: $X \to Y$ such that $f(x) \in \Gamma_B \phi(x)$ for every $x \in X$.*

As was pointed out in [**1**, p. 364], Theorem 1.1 remains true if B is any complete, metrizable locally convex space, but it is generally false if B is not metrizable. We can, however, prove the following generalization of Theorem 1.1.

THEOREM 1.2. *Let X be paracompact, and M a metrizable subset of a complete*[2] *locally convex space E. Let ϕ: $X \to 2^M$ be l.s.c. and such that, for some metric on M, every $\phi(x)$ is complete. Then there exists a continuous f: $X \to E$ such that $f(x) \in \Gamma_E \phi(x)$ for every $x \in X$.*

Theorem 1.2 was proved in [**3**] under the stronger assumption that X is metrizable. While that was sufficient for the applications in [**3**], and probably for most other applications, it did not generalize Theorem 1.1, and was therefore never entirely satisfying. In this

Received by the editors May 23, 1966.

[1] Supported by NSF Grant 11-5020.

[2] It suffices if $\Gamma_E K$ is compact for every compact $K \subset M$.