SPECTRAL DECOMPOSITION OF QUASI-MONTEL SPACES

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In [8] the author showed that Montel spaces have the property that all regular Borel spectral measures with values in their continuous-linear-transformation algebras are necessarily purely atomic. The purpose of this note is to make the observation that by virtue of a theorem of Bartle, Dunford and Schwartz [1] and Grothendieck [3], this property is shared by a significantly larger class of locally convex spaces, namely the quasi-Montel spaces of K. Kera [5]. This class includes the classical Banach space l^1 and its subspaces and the "gestufte Räume" of Köthe and their subspaces [6]. The result given in this note also has consequences, which we shall mention briefly, in the study of the singular operators of Kantorovitz [4].

Let $E[\mathfrak{T}]$ be a locally convex topological vector space, which will be assumed to be boundedly complete; recall that a *spectral measure triple* in $\mathfrak{L}(E)$ is a triple $(X, \mathfrak{S}, \mu)$ where X is a set, $\mathfrak{S}$ is a σ-algebra of subsets of X, and μ: $\mathfrak{S}\rightarrow\mathfrak{L}(E)$ is an $\mathfrak{L}(E)$-valued set function which is countably additive in the weak operator topology and for which $\mu(X)=1$ (the identity transformation) and for any δ, $\epsilon\in\mathfrak{S}$, $\mu(\delta\cap\epsilon)=\mu(\delta)\cdot\mu(\epsilon)$. $(X, \mathfrak{S}, \mu)$ is said to be *equicontinuous* if the values of μ on $\mathfrak{S}$ are, and to be *Baire* or *Borel* if X is a compact Hausdorff space and $\mathfrak{S}=\mathfrak{B}_0$ or $\mathfrak{B}$, its σ-algebras of Baire or Borel sets respectively. A Borel spectral measure triple $(X, \mathfrak{B}, \mu)$ is said to be *regular* if $\langle\mu(\cdot)x, x'\rangle$ is a regular Borel measure for each $x\in E$ and $x'\in E'$ (cf. [8, Proposition 3.18]). A *point atom* of a Borel measure is a point $\xi\in X$ with $\mu(\{\xi\})\neq 0$. For $x\in E$, the *cyclic subspace* and *real cyclic subspace generated by* x, denoted by $\mathfrak{M}(x)$ and $\mathfrak{M}_R(x)$ respectively, are the smallest closed subspace and closed real subspace of E respectively which contain $\{\mu(\delta)x\}_{\delta\in\mathfrak{S}}$. [8, Proposition 3.15] shows that both $\mathfrak{M}_R(x)$ and $\mathfrak{M}(x)$ are complete locally convex spaces when E is boundedly complete in $\mathfrak{T}$, [8, Proposition 3.13 et seq.] that $\mathfrak{M}_R(x)$ is a complete vector lattice when ordered by taking its positive cone to be the closed convex cone generated by $\{\mu(\delta)x\}_{\delta\in\mathfrak{S}}$, and also that $\mathfrak{M}(x)=\mathfrak{M}_R(x)\oplus i\mathfrak{M}_R(x)$.

It is easy to see that only small modifications of the proof of [8, Theorem 4.1] suffice to yield a proof of the slightly stronger-looking

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1. Proposition. *Let $E[\mathfrak{T}]$ be a boundedly complete locally convex space, $(X, \mathfrak{B}_0, \mu)$ an equicontinuous Baire spectral measure triple in $\mathfrak{L}(E)$, and $(X, \mathfrak{B}, \tilde{\mu})$ its unique extension to a regular Borel spectral measure (see [8, Proposition 3.18]). Suppose that for each idempotent $e\neq 0$ in the strong closure of $\{\mu(\delta)\}_{\delta\in\mathfrak{B}_0}$ there exists a nonzero $x\in eE$ for which the order interval $[-x, x]\subseteq\mathfrak{M}_R(x)$ is compact. Then $(X, \mathfrak{B}, \tilde{\mu})$ possesses atoms, hence point atoms, and their supremum (in the sense of [8, Proposition 3.17]) is the identity element of $\mathfrak{L}(E)$.*

Indeed, taking $e=1$ one sees that there does exist a nonzero $x\in E$ for which $[-x, x]\subseteq\mathfrak{M}_R(x)$ is compact, whence $\tilde{\mu}$ possesses some point atoms as in the proof of Theorem 4.1 of [8]. Letting f be the supremum (in the sense of [8, Proposition 3.17]) of the projections $\{\tilde{\mu}(\{\xi\})\}_{\xi\in X}$, we then see that unless $f=1$ the hypothesis of this proposition can be applied to the idempotent $e=1-f$, yielding a nonzero $y\in eE$ for which the order interval $[-y, y]\subseteq\mathfrak{M}_R(y)\subseteq eE$ is compact, and thus as in [8] giving a point atom ξ for which $\tilde{\mu}(\{\xi\})\cdot(1-f)\neq 0$, which is absurd.

For a given equicontinuous spectral measure triple $(X, \mathfrak{S}, \mu)$, let M and M_R denote respectively the algebras of bounded complex- and real-valued $\mathfrak{S}$-measurable functions on X.

2. Lemma. *Let $E[\mathfrak{T}]$ be a boundedly complete locally convex space and $(X, \mathfrak{S}, \mu)$ an equicontinuous spectral measure triple in $\mathfrak{L}(E)$. Then for each $x\in E$ the interval $[-x, x]\subseteq\mathfrak{M}_R(x)$ is the closure of the set $\{\int f d\mu(x) \mid f\in M_R, |f|\leqq 1\}$.*

Proof. Since the interval is closed it will suffice to show that the latter set is dense in it. Let q be any seminorm on E compatible with $(X, \mathfrak{S}, \mu)$ in the sense of [8, Proposition 2.3 ff.]; then μ induces a spectral measure $\hat{\mu}_q$ on $\hat{E}_q=(E/q^{-1}[0])^\wedge$ for which the natural quotient map $z\to z_q$ of $E\to\hat{E}_q$ preserves the algebraic and lattice operations on cyclic subspaces [8, Lemma 3.12]. In particular, this quotient map sends $[-x, x]\subseteq\mathfrak{M}_R(x)$ into $[-x_q, x_q]\subseteq\mathfrak{M}_R(x_q)\subseteq\hat{E}_q$. Since $\hat{E}_q$ is a Banach space, for each $y\in[-x, x]$ there exists $f\in M_R$, $-1\leqq f\leqq 1$, with $y_q=\int f d\hat{\mu}_q(x_q)$ [8, Theorem 3.9]; in other words, $y_q=(\int f d\mu(x))_q$, or $q(y-\int f d\mu(x))=0$. Since there are enough compatible seminorms to generate $\mathfrak{T}$, this shows that $\{\int f d\mu(x) \mid f\in M_R, |f|\leqq 1\}$ is dense in $[-x, x]$.

Remark. It follows that in the event there exists a continuous norm on $\mathfrak{M}_R(x)$, one has $\{\int f d\mu(x) \mid f\in M_R, |f|\leqq 1\}=[-x, x]$.

Now suppose that one is given a boundedly complete space $E[\mathfrak{T}]$ which is a quasi-Montel space in the sense of [5], i.e., its weakly compact subsets are $\mathfrak{T}$-compact, and suppose $(X, \mathfrak{B}_0, \mu)$ is an equicontinuous Baire spectral measure triple in $\mathfrak{L}(E)$; let $(X, \mathfrak{B}, \tilde{\mu})$ be its regular Borel extension. Then for each $x \in E$ the mapping $f \rightarrow \int f d\mu(x)$ is a continuous linear mapping from $\mathfrak{C}_R(X)$ to $\mathfrak{M}_R(x)$ with the property that for each closed $\delta \subseteq X$ the linear functional on E' defined by $x' \rightarrow \lim_f \langle \int f d\mu(x), x' \rangle$, where f runs through the naturally downward-directed set $\{f \mid 0 \leqq f \in \mathfrak{C}_R(X), f \geqq \chi_\delta\}$, is $\sigma(E', E)$-continuous, namely is just $x' \rightarrow \langle \tilde{\mu}(\delta)x, x' \rangle$, because the measure $\tilde{\mu}$ is regular. Therefore by [**3**, Theorem 6][2] this mapping is weakly compact, i.e., the closure of $\{\int f d\mu(x) \mid f \in \mathfrak{C}_R(X), |f| \leqq 1\}$ is weakly compact, and since E is quasi-Montel $\mathfrak{T}$-compact, in E. But even the weak compactness of the mapping $f \rightarrow \int f d\mu(x)$ implies that it can be extended by weak continuity to a map from $\mathfrak{C}_R(X)'' \rightarrow \mathfrak{M}_R(x)$ which takes the unit ball of the former to the closure of $\{\int f d\mu(x) \mid f \in \mathfrak{C}_R(X), |f| \leqq 1\}$ in the latter. In particular, then, $\{\int f d\mu(x) \mid f\ \mathfrak{B}\text{-measurable}, |f| \leqq 1\}$ is contained in a compact set, and thus its closure, which by Lemma 2 above is $[-x, x]$, is $\mathfrak{T}$-compact. We have proved

3. Theorem. *If $E[\mathfrak{T}]$ is a boundedly complete quasi-Montel space, then every equicontinuous Baire spectral measure in $\mathfrak{L}(E)$ has purely atomic regular Borel extension, i.e., the regular Borel extension possesses point atoms and the supremum of the projections corresponding to those point atoms is the identity.*

Indeed, we have shown that the hypotheses of Proposition 1 are satisfied.

Moreover, since any equicontinuous σ-complete Boolean algebra of idempotents in $\mathfrak{L}(E)$ can be realized as the values of a Baire spectral measure on its Stone space, we have [**8**, Corollary 4.6] available on boundedly complete quasi-Montel spaces as well: the proof given in [**8**] uses only atomicity of $\tilde{\mu}$.

4. Corollary. *If E is a boundedly complete quasi-Montel space, then every equicontinuous σ-complete Boolean algebra in $\mathfrak{L}(E)$ has a purely atomic completion equal to its strong closure in $\mathfrak{L}(E)$.*

Examples of the spaces we have been considering are abundant: any perfect Köthe sequence space λ, equipped with its normal topology, is a complete quasi-Montel space [**6**, pp. 416 and 419]. Thus 3 and 4 above apply to any such λ or to any of its closed subspaces.

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[2] Or, alternatively, by representing the complete subspace $\mathfrak{M}_R(x)$ as a projective limit of Banach spaces, then observing that the induced maps $\mathfrak{C}_R(X) \rightarrow \mathfrak{M}_R(x) \rightarrow \mathfrak{M}_R(x_q)$ are weakly compact by [1, Theorem 3.2], whence the map $\mathfrak{C}_R(X) \rightarrow \mathfrak{M}_R(x)$ is also.

Furthermore, since perfect sequence spaces are weakly sequentially complete [**6**, p. 415], any homomorphism of a $\mathcal{C}(X)$, X compact Hausdorff, into $\mathcal{L}(\lambda)$ which sends the unit sphere of $\mathcal{C}(X)$ to an equicontinuous subset of $\mathcal{L}(\lambda)$ can be given by an equicontinuous Baire spectral measure as in [**7**], and that measure may then be extended to a uniquely determined regular Borel measure as in [**8**, Proposition 3.18]. In particular, any "gestufter Raum" of Köthe [**6**, p. 422] is complete, metrizable, separable and quasi-Montel in its normal topology, so any weakly continuous homomorphism of a $\mathcal{C}(X)$ into its linear-transformation algebra sends the unit sphere to an equicontinuous set and can be given by an equicontinuous regular Borel spectral measure, which must be purely atomic; similarly, for these spaces the hypothesis of equicontinuity for σ-complete Boolean algebras is automatically satisfied [**8**, Proposition 1.2], while completeness and σ-completeness are equivalent by separability and metrizability: any σ-complete Boolean algebra on such a space is complete and purely atomic with only countably many atoms. The simplest example of such a "gestufter Raum" is, of course, the classical Banach space l^1.

Theorem 3 takes the following form for operators which are "scalar" in the sense of Dunford [**2**] ("spectral" in the sense of [**7**]); the proof is the same as that of [**8**, Corollary 4.8].

5. COROLLARY. *Let $E[\mathfrak{T}]$ be boundedly complete and quasi-Montel, u a scalar operator with domain D_u and equicontinuous spectral measure ν (defined on the Borel sets of* **C**) *for which $u=\int z d\nu$. Then*

$$ux = \int z d\nu(x) = \sum_{\lambda \in \pi(u)} \lambda \nu(\{\lambda\})x$$

for every $x \in D_u$ ($\pi(u)$ denotes the point spectrum of u).

More generally, it is not difficult to see that every operator with real spectrum on a quasi-Montel Banach space which is spectral of finite type n in the sense of Dunford is a singular operator of class C^n in the sense of Kantorovitz [**4**, Definition 3.9]. Indeed, the representation

$$T(f) = \sum_{j=0}^{n} \int f^{(j)}(s) d\left[\frac{N^j}{j!} E(s)\right] \qquad [4, \text{p. } 211]$$

for the C^n-operational calculus of T with spectral measure E and nilpotent part N shows that the measures

$$\mu_j(\cdot \mid x, x') = (1/j!)\langle NE(\cdot)x, (N')^{j-1}x'\rangle$$

for $j \geqq 1$ are purely atomic, thus *a fortiori* singular. A converse proposition, i.e., that singular operators on these spaces are spectral of finite type, would follow in the case of l^1 from a strengthened version of [**4**, Lemma 3.10] with the hypothesis of reflexivity replaced by that of weak sequential completeness.

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