A CLASS OF BANACH ALGEBRAS WITH A UNIQUE NORM TOPOLOGY[1]

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The purpose of this paper is to produce a class of nonsemisimple Banach algebras with a unique norm topology. That is, a class of Banach algebras such that each member B of the class has the property that any two Banach algebra norms on B are equivalent (cf. [**9**, Chapter II]). (Throughout this paper, an algebra will denote any commutative algebra over the complex field C which possesses an identity e.) To do this, we investigate an algebraic extension of a semisimple algebra A which is similar to the Arens-Hoffman extension of a normed algebra (cf. [**1**]). We let $\alpha(x)$ be a monic polynomial in $A[x]$, the algebra of all polynomials in the indeterminate x with coefficients in A, and denote by $(\alpha(x))$ the principal ideal in $A[x]$ generated by $\alpha(x)$. If $B=A[x]/(\alpha(x))$ is a Banach algebra with respect to some norm, then A is a normed algebra with respect to the norm on B restricted to A and we ask whether or not A is a closed subalgebra of B. If for any Banach algebra norm on B, A is a closed subalgebra of B, then we show that B has a unique norm topology.

The main result of this paper is that if A is a regular (in the language of [**9**], completely regular), semisimple Banach algebra and if $\alpha(x)$ is any monic polynomial in $A[x]$, then $B=A[x]/(\alpha(x))$ has a unique norm topology, where B is the Arens-Hoffman extension of A. (See below for a discussion of the Arens-Hoffman extension.) An example is given which shows that the condition of semisimplicity is essential. It is an open question whether or not the main result is true if A is not a regular Banach algebra.

In the event that A is a semisimple Banach algebra, $\alpha(x)$ is a monic polynomial in $A[x]$ and $B=A[x]/(\alpha(x))$ is semisimple, then by [**9**, Corollary 2.5.18], B has a unique norm topology. Therefore our results have meaning only if B is not semisimple. R. Arens and K. Hoffman have shown that if the discriminant d of $\alpha(x)$ is not a zero divisor in A, then B is semisimple [**1**, Theorem 4.3]. (See [**1**, p. 207] for the definition of d.) The converse of this latter result is also valid

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and is proved by J. Lindberg [6, Corollary 9.3]. Therefore B is not semisimple if and only if d is either zero or a divisor of zero.

Unless otherwise stated, A will denote a semisimple algebra, $\alpha(x)$ a monic polynomial in $A[x]$, and $B=A[x]/(\alpha(x))$. We will denote the coset $x+(\alpha(x))$ in B by $\mathfrak{x}$ and the coset $a+(\alpha(x))$ by a, for each $a\in A$. Thus any element $b\in B$ can be uniquely expressed in the form $b=\sum_{i=0}^{n-1} a_i\mathfrak{x}^i$, where n is the degree of $\alpha(x)$ and $a_0, a_1, \cdots, a_{n-1}$ are elements in A.

We begin by presenting a brief discussion of the Arens-Hoffman extension. If A is a normed algebra (not necessarily semisimple), and $\alpha(x)$ is a monic polynomial in $A[x]$, Arens and Hoffman have shown in [1] that $B=A[x]/(\alpha(x))$ possesses a family of norms each of which makes B into a normed algebra with the property that the natural embedding of A into B is an isometry as well as an isomorphism of A onto a closed subalgebra of B. Furthermore, B is a Banach algebra whenever A is a Banach algebra. If $\alpha(x)=x^n+\sum_{i=0}^{n-1}\alpha_i x^i$, then the family of norms is given by

$$\left\|\sum_{i=0}^{n-1} a_i\mathfrak{x}^i\right\| = \sum_{i=0}^{n-1}\|a_i\|t^i,$$

where $\|\cdot\|$ is the norm on A and t is any positive number such that $\sum_{i=0}^{n-1}\|\alpha_i\|t^i\leqq t^n$. Since there always exists a t with the above property, we shall assume $t=1$. We will refer to B as an Arens-Hoffman extension of A and call the norm

$$\left\|\sum_{i=0}^{n-1} a_i\mathfrak{x}^i\right\| = \sum_{i=0}^{n-1}\|a_i\|t^i,$$

for any appropriate t, an Arens-Hoffman norm.

We denote by Φ_A the carrier space of the normed algebra A and by $\hat{A}$ the Gelfand representation of A (cf. [9, Chapter III]). If $h\in\Phi_A$, we view x as an indeterminate over C as well as an indeterminate over A and let $\alpha_h(x)$ denote the polynomial $\sum_{i=0}^{n-1}\hat{\alpha}_i(h)x^i$ in $C[x]$. Arens and Hoffman have shown that if A is a normed algebra and if $\alpha(x)$ is a monic polynomial in $A[x]$, then $\Phi_B(B=A[x]/(\alpha(x)))$ is identifiable with the set $\{(h,\lambda)\in\Phi_A\times C:\alpha_h(\lambda)=0\}$ together with the relative topology of $\Phi_A\times C$ [1, Theorem 4.2]. The Arens-Hoffman extension has been studied by G. A. Heuer in [4], J. A. Lindberg in [6], [7], and [8], and by Heuer and Lindberg in [5].

We now return to our assumption that A is a semisimple algebra. Our first result shows that if B is a Banach algebra, then there exists a norm on A with respect to which A is a Banach algebra.

LEMMA 1. *Assume B is a Banach algebra with norm $\|\cdot\|$. Then there is a norm $\|\cdot\|'$ on A with respect to which A is a Banach algebra. Also, $\|\cdot\|'$ is majorized by the norm in B restricted to A.*

PROOF. Let R be the radical of B. Since $\hat{B}$ is isomorphic to B/R, $\hat{B}$ is a Banach algebra with respect to the quotient norm $\|\hat{b}\|_Q = \inf_{r\in R} \|b+r\|$. We will show that $\hat{A}$ is a $\|\cdot\|_Q$-closed subalgebra of $\hat{B}$. Assume $\{\hat{a}_m\}$ is a Cauchy sequence in $\hat{A}$. Then there is an element $b\in B$ such that $\|\hat{a}_m-\hat{b}\|_Q\to 0$ as $m\to\infty$. Since $\|\cdot\|_Q$ majorizes the sup norm on B, $\|a_m-b\|_\infty\to 0$ as $m\to\infty$ and thus $\hat{b}$ is constant on each fiber of Φ_B. For each $h\in\Phi_A$, denote the roots of $\alpha_h(x)=0$ by $\lambda_1(h)$, $\lambda_2(h), \cdots, \lambda_n(h)$, where each distinct root is repeated according to its multiplicity and n is the degree of $\alpha(x)$. Also, let $s_1(h), s_2(h), \cdots, s_n(h)$ be the elementary symmetric functions of $\lambda_1(h), \lambda_2(h), \cdots, \lambda_n(h)$. Since

$$\alpha_h(x) = \prod_{i=1}^{n} (x - \lambda_i(h)) = x^n + \sum_{j=1}^{n} (-1)^j s_j(h) x^{n-j}$$

for each $h\in\Phi_A$,

$$\hat{\alpha}(x) = x^n + \sum_{i=0}^{n-1} \hat{\alpha}_i x^i = x^n + \sum_{j=1}^{n} (-1)^j s_j x^{n-j}.$$

But this means each s_j is an element of $\hat{A}$ since $\hat{\alpha}$ is a polynomial over $\hat{A}$.

Now define a function f on Φ_A by $f(h)=\sum_{j=1}^{n} \hat{b}(h, \lambda_j(h))$ for each $h\in\Phi_A$. Clearly, $f\in C(\Phi_A)$. We will show that $f\in\hat{A}$. We view f as a function on Φ_B by writing $f(h, \lambda_i(h))=f(h)$ for any $(h, \lambda_i(h))\in\Phi_B$. Thus f is a symmetric function in the $\lambda_j(h)$'s and can be expressed as a polynomial in the s_j's with coefficients in $\hat{A}$. Since each $s_j\in\hat{A}$, $f\in\hat{A}$. (This result is a special case of [**8**, Lemma 1.2].)

We have already shown that $\hat{b}$ is constant on the fibers of Φ_B and thus $f(h)=n\hat{b}(h, \lambda_j(h))$ for any $j=1, 2, \cdots, n$. Therefore $\hat{b}\in\hat{A}$. We have thus shown that $\hat{A}$ is a closed subalgebra of $\hat{B}$.

Now define $\|a\|'=\|a\|_Q$ for all $a\in A$. Since A is isomorphic to $\hat{A}$, $\|\cdot\|'$ defines a norm on A with respect to which A is a Banach algebra. Furthermore, $\|a\|'=\inf_{r\in R} \|a+r\| \leqq \|a\|$ and therefore the proof is complete.

We note that by [**9**, Corollary 2.5.18], $\|\cdot\|'$ is equivalent to any other Banach algebra norm on A.

THEOREM 2. *If A has the property that for any Banach algebra norm on B, A is a closed subalgebra of B, then B has a unique norm topology.*

PROOF. Assume $\|\cdot\|$ is a Banach algebra norm on B. By Lemma 1, A is a Banach algebra with norm $\|\cdot\|'$ and for each $a\in A$, $\|a\|'\leqq\|a\|$. Since A is a closed subalgebra of B, by [9, Corollary 2.5.18], there exists a constant K such that $\|a\|'\leqq\|a\|\leqq K\|a\|'$ for each $a\in A$. Let $\|\cdot\|_1$ denote the Arens-Hoffman norm on B which extends $\|\cdot\|'$ on A, i.e. $\|\sum_{i=0}^{n-1} a_i\mathfrak{x}^i\|_1=\sum_{i=0}^{n-1}\|a_i\|'$. To show B has a unique norm topology, we show that $\|\cdot\|$ is equivalent to $\|\cdot\|_1$. If $M=\max_{0\leq i\leq n-1}\|\mathfrak{x}^i\|$, then for any $a_0, a_1, \cdots, a_{n-1}$ in A,

$$\left\|\sum_{i=0}^{n-1} a_i\mathfrak{x}^i\right\| \leqq M\sum_{i=0}^{n-1}\|a_i\| \leqq MK\sum_{i=0}^{n-1}\|a_i\|' = MK\left\|\sum_{i=0}^{n-1} a_i\mathfrak{x}^i\right\|_1,$$

and, by the inverse mapping theorem, the two norms are equivalent. Therefore any norm on B is equivalent to the Arens-Hoffman norm and the proof is complete.

We now prove two conditions on A which are sufficient for A to be a closed subalgebra of B. In the following theorem, we view A as a function algebra. This involves no loss of generality since A is assumed to be a semisimple algebra. Thus if B is a Banach algebra, by Lemma 1, A is also a Banach algebra and therefore A is a semisimple Banach algebra.

THEOREM 3. *Let X be a compact Hausdorff space and let A be a regular subalgebra of $C(X)$ which contains the constant functions. If B is a Banach algebra with norm $\|\cdot\|$, then A is a closed subalgebra of B.*

PROOF. By Lemma 1, there exists a norm $\|\cdot\|'$ on A with respect to which A is a Banach algebra and $\|a\|'\leqq\|a\|$ for each $a\in A$. Since the identity mapping is an isomorphism of A into B, by [2, Theorem 3.7 and Corollary 3.9], there is a finite set $F=\{h_1, h_2, \cdots, h_k\}$ in Φ_A such that for any neighborhood V in Φ_A of F, there is a constant M_V such that $\|a\|\leqq M_V\|a\|'$ for all $a\in K(V)$, where $K(V)=\{c\in A: \hat{c}=0 \text{ on } V\}$ is the kernel of V. For any such V, let $u_V\in A$ be such that $\hat{u}_V=1$ on V.

To show A is a closed subalgebra of B, let $\{a_m\}$ be a Cauchy sequence in A with respect to $\|\cdot\|$. Thus there exists an element $b\in B$ such that $\|a_m-b\|\to 0$ as $m\to\infty$. But $\{a_m\}$ is also a Cauchy sequence with respect to $\|\cdot\|'$ and hence there exists an element $c\in A$ such that $\|a_m-c\|'\to 0$ as $m\to\infty$. For any neighborhood V in Φ_A of F, $a_m-a_mu_V \in K(V)$ for each $m=1, 2, \cdots,$ and thus $\|a_m-a_mu_V\| \leqq M_V\|a_m-a_mu_V\|'$. Therefore $\|a_m-a_mu_V-(c-cu_v)\|\to 0$ as $m\to\infty$ and hence $c-cu_V=b-bu_V$. Let $b_0, b_1, \cdots, b_{n-1}$ be in A such that $b=\sum_{i=0}^{n-1} b_i\mathfrak{x}^i$ where n is the degree of $\alpha(x)$. Then $c-cu_V=b_0-b_0u_V$

and for $i=1, 2, \cdots, n-1$, $b_i = b_i u_V$ which means $\hat{b}_i = 0$ on $\{h \in \Phi_A : \hat{u}_V(h) \neq 1\}$.

To show $b_i = 0$ for $i=1, 2, \cdots, n-1$, we first show $\hat{b}_i = 0$ on $\Phi_A - F$. If $h_0 \in \Phi_A - F$, let V be a closed neighborhood in Φ_A of F such that $h_0 \notin V$ and let $u_V \in A$ such that $\hat{u}_V = 1$ on V and $\hat{u}_V(h_0) = 0$. These choices are possible since Φ_A is compact Hausdorff and A is regular. Therefore $\hat{b}_i(h_0) = \hat{b}_i(h_0)\hat{u}_V(h_0) = 0$ for $i=1, 2, \cdots, n-1$, and since h_0 was arbitrary, $\hat{b}_i = 0$ on $\Phi_A - F$. We next show that each $\hat{b}_i = 0$ on F. Since each $\hat{b}_i$ is continuous on Φ_A, each point $h_j \in F$ such that $\hat{b}_i(h_j) \neq 0$ for some $i=1, 2, \cdots, n-1$ is an isolated point of Φ_A. By relabeling these points if necessary, we may and do assume that there exists an integer $k' \leqq k$ such that for each $j=1, 2, \cdots, k'$, there is an i such that $1 \leqq i \leqq n-1$ and $\hat{b}_i(h_j) \neq 0$. Thus for each $j=1, 2, \cdots, k'$, there exists an idempotent u_j in A such that $\hat{u}_j(h_j) = 1$ and $\hat{u}_j = 0$ on $\Phi_A - \{h_j\}$. For any such j, consider the ideal (u_j) in A generated by u_j. This ideal is one-dimensional and thus is closed with respect to both norms in A. Since $a_m u_j \in (u_j)$ for each m, $bu_j \in (u_j) \subset A$. But $bu_j = \sum_{i=0}^{n-1} b_i u_j \mathfrak{x}^i$ and thus for each $i=1, 2, \cdots, n-1$, $b_i u_j = 0$. Therefore each $\hat{b}_i(h_j) = 0$ and this means, since j was arbitrary, that each $\hat{b}_i$ is identically zero on F. Since A is semisimple, each $b_i = 0$ and therefore $b = b_0 \in A$. Thus A is a closed subalgebra of B and the proof is complete.

The techniques used in the above proof to show that each $\hat{b}_i$ is zero on F are used to prove the following theorem.

THEOREM 4. *Assume B is a Banach algebra with norm $\|\cdot\|$. If the isolated points of Φ_A are hull-kernel dense in Φ_A, then A is a closed subalgebra of B.*

PROOF. Let $I = \{h_\gamma : \gamma \in \Gamma\}$ be the isolated points in Φ_A. For each $\gamma \in \Gamma$, let e_γ be the idempotent in A such that $\hat{e}_\gamma(h_\gamma) = 1$ and $\hat{e}_\gamma = 0$ on $\Phi_A - \{h_\gamma\}$. Let $\{a_m\}$ be a Cauchy sequence in A and assume for $b = \sum_{i=0}^{n-1} b_i \mathfrak{x}^i \in B$ that $\|a_m - b\| \to 0$ as $m \to \infty$. For each γ, the ideal (e_γ) in A generated by e_γ is one dimensional and hence is closed in A with respect to the norm in B restricted to A. Since $a_m e_\gamma \in (e_\gamma)$ for all m, $be_\gamma \in (e_\gamma) \subset A$ and thus $b_i e_\gamma = 0$ for $i=1, 2, \cdots, n-1$. But this means that $\hat{b}_i(h_\gamma) = 0$ and, since γ was arbitrary, $b_i \in K(I)$ for $i=1, 2, \cdots, n-1$. Therefore $\hat{b}_i = 0$ on $HK(I)$, the hull of the kernel of I, which by assumption is Φ_A. Finally, since A is semisimple, $b_i = 0$ and hence $b = b_0 \in A$. This completes the proof.

We conclude with an example showing that the semisimplicity of A is essential in Theorem 3.

EXAMPLE. Let A_1 be any commutative Banach algebra without identity having the property that there exists a dense subalgebra A_0 of A_1 such that $A_1 = A_0 \oplus R$, where the radical of A_1 is R, and R is a principal ideal generated by an element r such that $r^2 = ra = 0$ for any $a \in A_0$. (See [2] or [3] for specific examples.) Thus A_0 is a Banach algebra with respect to the quotient norm $\|a\|_Q = \inf_{\gamma \in C} \|a + \gamma r\|$. Adjoin an identity e to A_1 and obtain $A = A_1 \oplus \{\lambda e\}$. Let $B = A[x]/(x^2) = A \oplus A\mathfrak{x}$ and define a norm on B as follows.

$$\|a+\gamma r+\lambda e+(b+\mu r+\nu e)\mathfrak{x}\|_1 = \|a+\mu r\| + |\gamma| + |\lambda| + \|b\|_Q + |\nu|$$

where $a, b \in A_0$. It is straightforward to verify that B is a Banach algebra with norm $\|\cdot\|_1$. If we restrict $\|\cdot\|_1$ to A, we obtain $\|a+\gamma r+\lambda e\|_1 = \|a\| + |\gamma| + |\lambda|$. Since A_0 is not a closed subalgebra of A_1, there exists a Cauchy sequence $\{a_m\}$ in A_0 such that for some $a+\mu r$ in A_1 with $\mu \neq 0$, $\|a_m - a - \mu r\| \to 0$ as $m \to \infty$. Thus $\{a_m - a\}$ is a Cauchy sequence in A which converges to $\mu r\mathfrak{x}$ with respect to $\|\cdot\|_1$. Therefore A is not a closed subalgebra of B with respect to $\|\cdot\|_1$.

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