AN INEQUALITY FOR CERTAIN SCHLICHT FUNCTIONS

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Let $S$ denote the classical family of schlicht functions $f$ on the unit disk $E$ which have the Taylor expansion $f(z) = z + \sum_{j=2}^{\infty} A_j z^j$. Recently Ozawa [5] used the Grunsky inequalities and an inequality of Jenkins to show that if $A_2 \geq 0$ then $\Re A_6 \leq 6$ with equality occurring only for the Koebe slit function $z/(1-z)^2$. In this note we shall show that if $A_j$ is real for $j \leq p$ then $\Re A_n \leq n$ for $n \leq 2p + 1$ with equality occurring only for one of the Koebe slit functions $z/(1 \pm z)^2$. This will be established by using a continuity argument to deduce from Jenkins’ General Coefficient Theorem [1] that the extremal functions for this coefficient problem have real coefficients.

Let $S_p = \{ f \in S : A_j \text{ is real for } j \leq p \}$, $S_\infty = \{ f \in S : A_j \text{ is real for every } j \}$. Set $V_{p,n} = \{ (\Re A_1, \ldots, \Re A_n) : f \in S_p \}$. Let $H_{n,\epsilon}$ ($\epsilon = \pm 1$) denote the metric space of symmetric pairs $(\Omega, g)$ defined as follows. First $\Omega = P(w)dw^2$ is a quadratic differential on the Riemann sphere $R$ of the canonical form

\begin{align}
(1) & \quad \overline{P}(\bar{w}) = P(w), \\
(2) & \quad P(w) = \alpha K \left[ \prod_{j=1}^{n} \frac{1}{w-a_j} \right] dw^2,
\end{align}

where $\alpha = \pm 1$, $K > 0$, $2 \leq s \leq n$, $0 \leq r \leq s-2$ (we adopt the convention that $\prod_{j=1}^{m} u_j = 1$ if $m < k$) and where $\alpha = \epsilon$ if $s = n$. Second $g \in S_\infty$. (Notice that $g \in S_\infty$ if and only if $g \in S$ and $g(z) = f(z)$.) Third $g$ and $\Omega$ are associated [3]. (In other words $g(E)$ is an admissible domain with respect to $\Omega$ in the sense of Jenkins.) Finally the metric $d$ on $H_{n,1} \cup H_{n,-1}$ is defined by the equation

$$d((\Omega_1, g_1), (\Omega_2, g_2)) = \sup \{ |P_1(w) - P_2(w)| : |w| = 1 \} + \sup \{ |g_1(z) - g_2(z)| : |z| = 1/2 \}.$$ 

The pairs of $H_{n,1}$, $H_{n,-1}$ will be denoted by $(\Omega_*, g_*), (\Omega**, g**) \text{ instead of } (\Omega, g)$. The coefficients of $g_*, g**$ will be denoted by $B_j, C_j$ instead of $A_j$.

**Theorem 1.** If $p \geq 1$ and $n \leq 2p$ then $V_{p,n}$ is homeomorphic to a closed ball in $R^{n-1}$, the real Euclidean space of $n-1$ dimensions. Every point of $\partial V_{p,n}$, the topological boundary of $V_{p,n}$, is taken by a unique slit function in $S_\infty$. Every point of $\text{int } V_{p,n}$, the interior of $V_{p,n}$, is taken

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by infinitely many bounded schlicht functions in $S_p$. In addition

$$V_{p,n} = \{ (B_1, \ldots, B_n): (\Omega_*, g_*) \in H_{n+1,1} \} = \{ (C_1, \ldots, C_n): (\Omega_**, g**) \in H_{n+1,-1} \}.$$ 

**Proof.** When $p=1$ the facts in Theorem 1 are either trivial or else follow immediately from well-known facts. Suppose Theorem 1 is true for $p \leq q - 1$. Then if $p = q$ and $n \leq 2q - 2$ the truth of Theorem 1 follows since $S_q \subset S_{q-1}$ and $V_{q,n} = V_{q-1,n}$. Therefore if $f \in S_q$ then $(\Omega_*, g_*)$, $(\Omega_**, g**)$ exist in $H_{2q-1,1}, H_{2q-1,-1}$ such that

$$(3) \quad (B_1, \ldots, B_2q-2) = (\Re A_1, \ldots, \Re A_{2q-2}) = (C_1, \ldots, C_{2q-2}).$$

If $\Omega_*$ or $\Omega_**$ has a pole at the origin of order less than 2q then by the induction hypothesis $g_* = f = g**$, $(\Re A_1, \ldots, \Re A_{2q-2}) \in \partial V_{q,2q-2}$ and hence $(\Re A_1, \ldots, \Re A_{2q-1}) \in \partial V_{q,2q-1}$. Therefore we may assume that the poles are of order 2q and $(\Re A_1, \ldots, \Re A_{2q-2}) \in \text{int} V_{q,2q-2}$ is taken by a bounded schlicht function $b \in S_p$.

Next we apply the General Coefficient Theorem in its current form [1] to $R$ the Riemann sphere and $\Omega$, $g(E), f \circ g^{-1}$, where $g = g_*$ or $g**, \Omega = \Omega_*$ or $\Omega_**$. Admissible homotopies into the identity are abundant, $g(E)$ is admissible by its definition, and $g \circ f^{-1}$ is admissible as well. In fact if $w$ is a suitable parameter representing the origin as the point at infinity then by (1), (2), (3) we obtain the expansions at infinity

$$P(1/w)(-1/w^2)^2 = \epsilon K \left[ w^{2q-4} + \sum_{j=1}^{\infty} \beta_j w^{2q-4-j} \right],$$

$$(5) \quad 1/(f \circ g^{-1})(1/w) = w + \sum_{j=q-1}^{\infty} a_j w^{-j},$$

where the $\beta_j$ are real and

$$a_j = (B_{j+2} - A_{j+2}) + (B_{q+1} - A_{q+1})^2 \max(0, j - 2q + 2)$$

$$+ \sum_{r=q+1}^{j+1} (B_r - A_r)n(v, j) \prod_{\mu=1}^{q} B_{\mu}e(\mu; v, j),$$

(or a corresponding expression with $C_r = B_r$) where $q-1 \leq j \leq 2q-1$ and $n(v; j)$, $e(\mu; v, j)$ are integers. Therefore taking $m = 2q$, $m-3 = 2q-3$, $k = (m-4)/2 = q-2$ and $a_{q-2} = 0$ in the General Coefficient Theorem we obtain

$$\Re \left\{ \epsilon \left[ a_{2q-3} + \sum_{j=1}^{q-3} \beta_j a_{2q-3-j} \right] \right\} \leq 0.$$
Consequently by (6) it follows from (7) that
\[ B_{2q-1} + \sum_{j=q+1}^{2q-2} b_j(B_j - \partial A_j) \leq \partial A_{2q-1} \leq C_{2q-1} + \sum_{j=q+1}^{2q-2} c_j(C_j - \partial A_j), \]
where \( b_j, c_j \) are real. By (3) we obtain
\[ (8) \quad B_{2q-1} \leq \partial A_{2q-1} \leq C_{2q-1}. \]
If equality occurs in (8) then it occurs in (7) and by the General Coefficient Theorem \( f \) must be at worst a translation of \( g \) along trajectories of \( \Omega \) in the \( |\Omega|^{1/2} \)-metric. But translations alter the value of \( A_2 \), and therefore \( f=g \). Since \((\partial A_1, \ldots, \partial A_{2q-2})\) belongs to a bounded schlicht function \( b \in S_q \) we know that \( B_{2q-1} < C_{2q-1} \). Finally the conformal isotopy \( t^{-1}f(tz) \) \((0 < t \leq 1, f \in S)\) which deforms \( f \) into the identity in \( S \) may be used to establish the remaining topological facts about \( V_{q,2q-1} \).

To prove the last statement in Theorem 1 for \( V_{q,2q-1} \) we use the existence of a continuous function \( \theta_e : V_{q,2q-1} \rightarrow H_{2q,\epsilon} \) with the following property. If \( \psi : H_{2q,\epsilon} \rightarrow V_{q,2q-1} \) \((\epsilon = \pm 1)\) is defined by setting \( \psi_1(\Omega_*, g_*) = (B_1, \ldots, B_{2q-1}) \), \( \psi_1(\Omega_*, g_*) = (C_1, \ldots, C_{2q-1}) \), then the restriction of \( \psi \circ \theta_e \) to \( \partial V_{q,2q-1} \) is homotopic to the identity. Consequently it follows from Brouwer's Fixed Point Theorem that \( \psi_e(\theta_e(V_{q,2q-1})) = V_{q,2q-1} \) and hence \( \psi_e(H_{2q+1,\epsilon}) = V_{q,2q-1} \) for \( \epsilon = \pm 1 \).

The induction step is completed by merely repeating this argument to obtain the facts in Theorem 1 for \( V_{q,2q} \). Q.E.D.

A map which is an extension of \( \theta_e \) to mixed coefficient regions is defined by a straightforward but rather lengthy process in [4]. The main feature of our construction is the use of the space of induced positive quadratic differentials \( \Omega \circ f \) on \( E \) and the canonical decompositions of \( \partial E \) into pairs of identified arcs introduced by Schaeffer and Spencer [6, Chapter VIII].

**Theorem 2.** If \( f \in S \), then
\[ \partial A_n \leq n \quad \text{for} \quad n \leq 2p + 1 \]
with equality occurring only for one of the Koebe slit functions \( z/(1 \pm z)^2 \).

**Proof.** If \( n \leq 2p \) then Theorem 2 follows immediately from Theorem 1 (see (8)) because of the truth of the Bieberbach conjecture for functions in \( S_\infty \). If \( f \in S_{2p+1} \) then by Theorem 1 there is an \((\Omega_*, g_*) \in H_{2p+1,1}\) such that
Applying the General Coefficient Theorem as in the proof of Theorem 1 to \( \Omega_{2p}, g_{2p}(E), f \circ g_{2p}^{-1} \) with \( m = 2p + 2, \ m - 3 = 2p - 1, \ k = (m - 4)/2 = p - 1 \) we obtain

\[
\Re \left\{ -\left[ a_{2p-1} + \sum_{j=1}^{p} \beta_j a_{2p-1-j} + \frac{(p-1)}{2} a_{p-1}^2 \right] \right\} \leq 0
\]

where the \( a_j \) are defined in (5), (6). Using (9) to simplify (10) we obtain

\[
\Re A_{2p+1} + (\Re A_{p+1})^2 (p+1)/2 \leq C_{2p+1}
\]

which implies our result. Q.E.D.

We note that (8) shows \( \Re(A_n) \geq -n \) if \( f \in S_p \) and \( n \leq 2p \). Also, note that to prove \( \Re A_6 \leq 6 \) we need both \( A_2, A_3 \) real whereas Ozawa needs only \( A_2 \geq 0 \).

We wish to mention that this proof was inspired by a remarkably short and elegant proof of Löwner's inequality \( |A_3| \leq 3 \) which was given by Jenkins [2]. One merely applies the General Coefficient Theorem [1] to the Schiffer quadratic differentials [7, p. 442, (20)] to show that the extremal functions for this problem are in \( S_\omega \).

REFERENCES

2. ———, oral communication.

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