AN INEQUALITY FOR CERTAIN SCHLICHT FUNCTIONS

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Let S denote the classical family of schlicht functions f on the unit disk E which have the Taylor expansion $f(z)=z+\sum_{j=2}^{\infty} A_j z^j$. Recently Ozawa [5] used the Grunsky inequalities and an inequality of Jenkins to show that if $A_2 \geqq 0$ then $\Re A_6 \leqq 6$ with equality occurring only for the Koebe slit function $z/(1-z)^2$. In this note we shall show that if A_j is real for $j \leqq p$ then $\Re A_n \leqq n$ for $n \leqq 2p+1$ with equality occurring only for one of the Koebe slit functions $z/(1 \pm z)^2$. This will be established by using a continuity argument to deduce from Jenkins' General Coefficient Theorem [1] that the extremal functions for this coefficient problem have real coefficients.

Let $S_p=\{f \in S: A_j \text{ is real for } j \leqq p\}$, $S_\infty=\{f \in S: A_j \text{ is real for every } j\}$. Set $V_{p,n}=\{(\Re A_1, \cdots, \Re A_n): f \in S_p\}$. Let $H_{n,\epsilon}$ $(\epsilon=\pm 1)$ denote the metric space of symmetric pairs (Ω, g) defined as follows. First $\Omega=P(w)dw^2$ is a quadratic differential on the Riemann sphere R of the canonical form

$$\overline{P}(\bar{w}) = P(w), \tag{1}$$

$$P(w) = \alpha K\left[\prod_{j=1}^{r}(w-a_j)/w^{s+1}\right]dw^2, \tag{2}$$

where $\alpha=\pm 1$, $K>0$, $2 \leqq s \leqq n$, $0 \leqq r \leqq s-2$ (we adopt the convention that $\prod_{j=k}^{m} u_j=1$ if $m<k$) and where $\alpha=\epsilon$ if $s=n$. Second $g \in S_\infty$. (Notice that $g \in S_\infty$ if and only if $g \in S$ and $\bar{g}(\bar{z})=f(z)$.) Third g and Ω are *associated* [3]. (In other words $g(E)$ is an admissible domain with respect to Ω in the sense of Jenkins.) Finally the metric d on $H_{n,1} \cup H_{n,-1}$ is defined by the equation

$$\begin{aligned} d((\Omega_1, g_1), (\Omega_2, g_2)) &= \sup\{|P_1(w)-P_2(w)| : |w|=1\} \\ &\quad + \sup\{|g_1(z)-g_2(z)| : |z|=1/2\}. \end{aligned}$$

The pairs of $H_{n,1}$, $H_{n,-1}$ will be denoted by (Ω_*, g_*), (Ω_{**}, g_{**}) instead of (Ω, g). The coefficients of g_*, g_{**} will be denoted by B_j, C_j instead of A_j.

THEOREM 1. *If $p \geqq 1$ and $n \leqq 2p$ then $V_{p,n}$ is homeomorphic to a closed ball in R^{n-1}, the real Euclidean space of $n-1$ dimensions. Every point of $\partial V_{p,n}$, the topological boundary of $V_{p,n}$, is taken by a unique slit function in S_∞. Every point of* int $V_{p,n}$, *the interior of $V_{p,n}$, is taken*

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by infinitely many bounded schlicht functions in S_p. *In addition*

$$V_{p,n} = \{(B_1, \cdots, B_n): (\Omega_*, g_*) \in H_{n+1,1}\} = \{(C_1, \cdots, C_n): (\Omega_{**}, g_{**}) \in H_{n+1,-1}\}.$$

PROOF. When $p=1$ the facts in Theorem 1 are either trivial or else follow immediately from well-known facts. Suppose Theorem 1 is true for $p \leqq q-1$. Then if $p=q$ and $n \leqq 2q-2$ the truth of Theorem 1 follows since $S_q \subset S_{q-1}$ and $V_{q,n} = V_{q-1,n}$. Therefore if $f \in S_q$ then (Ω_*, g_*), (Ω_{**}, g_{**}) exist in $H_{2q-1,1}$, H_{2q-1-1} such that

$$(3) \quad (B_1, \cdots, B_{2q-2}) = (\mathfrak{R}A_1, \cdots, \mathfrak{R}A_{2q-2}) = (C_1, \cdots, C_{2q-2}).$$

If Ω_* or Ω_{**} has a pole at the origin of order less than $2q$ then by the induction hypothesis $g_* = f = g_{**}$, $(\mathfrak{R}A_1, \cdots, \mathfrak{R}A_{2q-2}) \in \partial V_{q,2q-2}$ and hence $(\mathfrak{R}A_1, \cdots, \mathfrak{R}A_{2q-1}) \in \partial V_{q,2q-1}$. Therefore we may assume that the poles are of order $2q$ and $(\mathfrak{R}A_1, \cdots, \mathfrak{R}A_{2q-2}) \in \operatorname{int} V_{q,2q-2}$ is taken by a bounded schlicht function $b \in S_p$.

Next we apply the General Coefficient Theorem in its current form [1] to R the Riemann sphere and Ω, $g(E)$, $f \circ g^{-1}$, where $g = g_*$ or g_{**}, $\Omega = \Omega_*$ or Ω_{**}. Admissible homotopies into the identity are abundant, $g(E)$ is admissible by its definition, and $g \circ f^{-1}$ is admissible as well. In fact if w is a suitable parameter representing the origin as the point at infinity then by (1), (2), (3) we obtain the expansions at infinity

$$(4) \qquad P(1/w)(-1/w^2)^2 = \epsilon K\left[w^{2q-4} + \sum_{j=1}^{\infty} \beta_j w^{2q-4-j}\right],$$

$$(5) \qquad 1/(f \circ g^{-1})(1/w) = w + \sum_{j=q-1}^{\infty} a_j w^{-j},$$

where the β_j are real and

$$(6) \qquad \begin{aligned} a_j = {} & (B_{j+2} - A_{j+2}) + (B_{q+1} - A_{q+1})^2 \max(0, j - 2q + 2) \\ & + \sum_{\nu=q+1}^{j+1} (B_\nu - A_\nu) n(\nu, j) \prod_{\mu=1}^{q} B_\mu e(\mu; \nu, j), \end{aligned}$$

(or a corresponding expression with $C_\nu = B_\nu$) where $q-1 \leqq j \leqq 2q-1$ and $n(\nu; j)$, $e(\mu; \nu, j)$ are integers. Therefore taking $m = 2q$, $m-3 = 2q-3$, $k = (m-4)/2 = q-2$ and $a_{q-2} = 0$ in the General Coefficient Theorem we obtain

$$(7) \qquad \mathfrak{R}\left\{\epsilon\left[a_{2q-3} + \sum_{j=1}^{q-2} \beta_j a_{2q-3-j}\right]\right\} \leqq 0.$$

Consequently by (6) it follows from (7) that

$$B_{2q-1} + \sum_{j=q+1}^{2q-2} b_j(B_j - \Re A_j) \leqq \Re A_{2q-1}$$
$$\leqq C_{2q-1} + \sum_{j=q+1}^{2q-2} c_j(C_j - \Re A_j),$$

where b_j, c_j are real. By (3) we obtain

$$B_{2q-1} \leqq \Re A_{2q-1} \leqq C_{2q-1}. \tag{8}$$

If equality occurs in (8) then it occurs in (7) and by the General Coefficient Theorem f must be at worst a translation of g along trajectories of Ω in the $|\Omega^{1/2}|$-metric. But translations alter the value of A_2, and therefore $f=g$. Since $(\Re A_1, \cdots, \Re A_{2q-2})$ belongs to a bounded schlicht function $b \in S_q$ we know that $B_{2q-1} < C_{2q-1}$. Finally the conformal isotopy $t^{-1}f(tz)$ $(0 < t \leqq 1, f \in S)$ which deforms f into the identity in S may be used to establish the remaining topological facts about $V_{q,2q-1}$.

To prove the last statement in Theorem 1 for $V_{q,2q-1}$ we use the existence of a continuous function θ_ϵ: $V_{q,2q-1} \to H_{2q,\epsilon}$ with the following property. If $\psi_\epsilon: H_{2q,\epsilon} \to V_{q,2q-1}$ $(\epsilon = \pm 1)$ is defined by setting $\psi_1(\Omega_*, g_*) = (B_1, \cdots, B_{2q-1})$, $\psi_{-1}(\Omega_{**}, f_{**}) = (C_1, \cdots, C_{2q-1})$, then the restriction of $\psi_\epsilon \circ \theta_\epsilon$ to $\partial V_{q,2q-1}$ is homotopic to the identity. Consequently it follows from Brouwer's Fixed Point Theorem that $\psi_\epsilon(\theta_\epsilon(V_{q,2q-1})) = V_{q,2q-1}$ and hence $\psi_\epsilon(H_{2q+1,\epsilon}) = V_{q,2q-1}$ for $\epsilon = \pm 1$.

The induction step is completed by merely repeating this argument to obtain the facts in Theorem 1 for $V_{q,2q}$. Q.E.D.

A map which is an extension of θ_ϵ to mixed coefficient regions is defined by a straightforward but rather lengthy process in [**4**]. The main feature of our construction is the use of the space of induced positive quadratic differentials $\Omega \circ f$ on E and the canonical decompositions of ∂E into pairs of *identified* arcs introduced by Schaeffer and Spencer [**6**, Chapter VIII]

THEOREM 2. *If $f \in S_p$ then*

$$\Re A_n \leqq n \quad \textit{for } n \leqq 2p + 1$$

with equality occurring only for one of the Koebe slit functions $z/(1 \pm z)^2$.

PROOF. If $n \leqq 2p$ then Theorem 2 follows immediately from Theorem 1 (see (8)) because of the truth of the Bieberbach conjecture for functions in S_∞. If $f \in S_{2p+1}$ then by Theorem 1 there is an $(\Omega_{**}, g_{**}) \in H_{2p+1,-1}$ such that

$$(9) \qquad (\mathfrak{R}A_1, \cdots, \mathfrak{R}A_{2p}) = (C_1, \cdots, C_{2p}).$$

Applying the General Coefficient Theorem as in the proof of Theorem 1 to Ω_{**}, $g_{**}(E)$, $f \circ g_{**}^{-1}$ with $m=2p+2$, $m-3=2p-1$, $k=(m-4)/2 = p-1$ we obtain

$$(10) \qquad \mathfrak{R}\left\{-\left[a_{2p-1} + \sum_1^p \beta_j a_{2p-1-j} + \frac{(p-1)}{2} a_{p-1}^2\right]\right\} \leqq 0$$

where the a_j are defined in (5), (6). Using (9) to simplify (10) we obtain

$$(11) \qquad \mathfrak{R}A_{2p+1} + (\mathfrak{I}A_{p+1})^2(p+1)/2 \leqq C_{2p+1}$$

which implies our result. Q.E.D.

We note that (8) shows $\mathfrak{R}(A_n) \geqq -n$ if $f \in S_p$ and $n \leqq 2p$. Also, note that to prove $\mathfrak{R}A_6 \leqq 6$ we need both A_2, A_3 real whereas Ozawa needs only $A_2 \geqq 0$.

We wish to mention that this proof was inspired by a remarkably short and elegant proof of Löwner's inequality $|A_3| \leqq 3$ which was given by Jenkins [2]. One merely applies the General Coefficient Theorem [1] to the Schiffer quadratic differentials [7, p. 442, (20)] to show that the extremal functions for this problem are in S_∞.

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