POLYNOMIAL APPROXIMATION OF FUNCTIONS ANALYTIC IN A DISK[1]

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Introduction. We consider the set H of functions analytic in the open unit disk and put $\|f\|_r=\sup\{|f(z)|: |z|<r\}$, $0<r\leqq 1$. Let $H_M=\{f\in H: \|f\|_1\leqq M\}$, $M>0$. P_n $(n\geqq 0)$ is the set of polynomials of degree not exceeding n while $P_{-1}=\{0\}$. The quantity

$$E_n(f, r) = \inf\{\|f - p_n\|_r: p_n \in P_n\} \qquad (0 < r \leqq 1, n \geqq -1)$$

is called the *degree of approximation of f by polynomials of degree not exceeding n in* $|z|<r$. For $A\subset H$,

$$E_n(A, r) = \sup\{E_n(f, r): f \in A\}, \qquad 0 < r \leqq 1,$$

is the degree of approximation of the class A. The object of this note is to determine $E_n(A, r)$ for many classes A. For the class $B^{(p)}=\{f\in H: f^{(p)}\in H_1\}$, $p=1, 2, 3 \cdots$. Babenko [1] has obtained

$$(1) \quad E_{n-1}(B^{(p)}, r) = \frac{r^n}{n(n-1)\cdots(n-p+1)}, \qquad n \geqq p, 0 < r \leqq 1.$$

This result cannot be called completely unexpected, since there are earlier theorems in approximation theory which relate a bounded pth derivative of f to $E_n(f)$ by a formula similar to (1) (compare Favard's theorem).

We propose a new approach, based on Babenko's, yet simpler, and more general. It is close to some classical proofs, in particular to the proof of Favard's theorem in [2]. Babenko's theorem is extended to nonintegral p, another differential operator is considered, and many other classes yield up their degrees of approximation. The main theorem explains how to estimate $E_n(f, r)$ by a suitable factorization of the Taylor coefficients of f.

We will work with the product in H:

$$(2) \quad h * g(z) = \sum_{k=0}^{\infty} a_k b_k z^k \quad \text{where} \quad h(z) = \sum_{k=0}^{\infty} b_k z^k, \quad g(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad h, g \in H.$$

Then also $h * g\in H$. Put

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$$A * g = \{f : f = h * g,\ h \in A\}. \tag{3}$$

The aim is to compute the degree of approximation $E_{n-1}(H_1 * g, r)$ for certain g. Since $E_{n-1}(f, r) = E_{n-1}(f + p_{n-1}, r)$ for each $p_{n-1} \in P_{n-1}$, we may assume $g(z) = \sum_{k=n}^{\infty} a_k z^k$ when convenient. The sequence $(a_k)_{k \geqq n}$, where n is a nonnegative integer will be called *positive* when the following series converges and

$$G(r, \theta) = \tfrac{1}{2} a_n + \sum_{k=1}^{\infty} a_{n+k} r^k \cos k\theta \geqq 0 \quad \text{for } 0 \leqq r < 1,\ \theta \text{ real}. \tag{4}$$

In this case, $g(z) = \sum_{k=n}^{\infty} a_k z^k$ belongs to H.

The main result.

THEOREM. *Let $(a_k)_{k \geqq n}$ be positive and $f = g * h$ with $h(z) = \sum_{k=0}^{\infty} b_k z^k \in H_1$ and $g(z) = \sum_{k=n}^{\infty} a_k z^k$. Then, for each $0 < R \leqq 1$, we have*

$$\|f + p_{n-1}\|_R \leqq a_n R^n,$$

$$p_{n-1}(z) = p_{n-1}(R, z) = \sum_{k=0}^{n-1} b_k a_{2n-k} R^2 (n - k) z^k. \tag{5}$$

Further, $f_0(z) = a_n z^n$ is extremal:

$$f_0 \in H_1 * g \quad \text{and} \quad E_{n-1}(f_0, R) = a_n R^n.$$

Hence

$$E_{n-1}(H_1 * g, R) = a_n R^n. \tag{6}$$

For the proof we need the following

LEMMA. *Let $(A_k)_{k \geqq n}$ be a positive sequence for which $\sum_{k=n}^{\infty} |A_k| < +\infty$. Then*

$$A_n = \min \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=-\infty}^{\infty} A_k e^{-ik\theta} \right| d\theta : \sum_{k=-\infty}^{n-1} |A_k| < +\infty \right\}. \tag{7}$$

To prove this, put

$$\Phi(\theta) = \sum_{k=-\infty}^{\infty} A_k e^{-ik\theta}.$$

Then

$$\int_0^{2\pi} |\Phi| \, d\theta \geqq \left| \int_0^{2\pi} e^{in\theta} \Phi(\theta) \, d\theta \right| = 2\pi A_n$$

using the absolute convergence. On the other hand, selecting

$A_k = A_{2n-k}$, $k \leqq n-1$, we have $\sum_{k=-\infty}^{n-1} |A_k| < \infty$ and

$$\Phi_0(\theta) = \sum_{k=-\infty}^{\infty} A_k e^{-ik\theta} = 2e^{-in\theta}\left(\frac{1}{2} A_n + \sum_{k=1}^{\infty} A_{n+k} \cos k\theta\right)$$
$$= 2e^{-in\theta} G(\theta, 1)$$

so that

$$\int_0^{2\pi} |\Phi_0| \, d\theta = 2\int_0^{2\pi} G(\theta, 1)\, d\theta = 2\pi A_n.$$

Turning to the proof of the theorem, let $|z| = R < \rho < 1$. We write the product (2) as an integral

$$f(z) = h * g(z) = \sum_{k=n}^{\infty} a_k b_k z^k = \frac{1}{2\pi i}\int_{|t|=\rho} h(t) g(z/t) t^{-1}\, dt.$$

If $r = R/\rho$ and

$$\sum_{k=-\infty}^{-1} |\xi_k| \, r^{|k|} < \infty, \tag{8}$$

then

$$f(z) + \sum_{k=0}^{k-1} \xi_k b_k z^k$$
$$= \frac{1}{2\pi i}\int_{|t|=\rho} h(t)\left[\sum_{k=-\infty}^{-1} \xi_k \left(\overline{\frac{z}{t}}\right)^{-k} + \sum_{k=0}^{n-1} \xi_k \left(\frac{z}{t}\right)^{k} + \sum_{k=n}^{\infty} a_k \left(\frac{z}{t}\right)^{k}\right]\frac{dt}{t}$$

due to the absolute convergence of the series (8) and the analyticity of h. Now letting $z/t = re^{-i\theta}$, and taking absolute values, we obtain

$$\left| f(z) + \sum_{k=0}^{n-1} \xi_k b_k z^k \right| \leqq \frac{i}{2\pi}\int_0^{2\pi} |\Phi| \, d\theta, \quad \Phi(\theta) = \sum_{k=-\infty}^{\infty} A_k e^{-ik\theta},$$
$$A_k = a_k r^k,\ k \geqq n,\ A_k = \xi_k r^{|k|},\ k \leqq n-1.$$

Hence

$$E_{n-1}(f, R) \leqq \left\| f + \sum_{k=0}^{n-1} \xi_k b_k u^k \right\|_R \leqq \frac{1}{2\pi}\int_0^{2\pi} |\Phi| \, d\theta$$

($u^k(z) = z^k$) for all such Φ. According to the lemma, we can minimize the right side by the choice $\xi_k r^{|k|} = a_{2n-k} r^{2n-k}$, $k \leqq n-1$, to obtain

$$E_{n-1}(f, R) \leqq \left\| f + \sum_{k=0}^{n-1} a_{2n-k} b_k r^{2(n-k)} u^k \right\|_R \leqq a_n r^n$$

for each ρ, $R<\rho<1$. Now letting $\rho\to 1$, we get the first part of the theorem. Since $f_0=u^n * g$, $u^n\in H_1$, we have $f_0\in H_1 * g$. From Rouché's theorem we conclude $\|f_0+p_{n-1}\|_R\leqq a_nR^n$ for all $p_{n-1}\in P_{n-1}$, so that $E_{n-1}(f_0, R)=a_nR^n$.

Applications. First some remarks on positive sequences $\alpha_k=a_{n+k}$, $k=0, 1, \cdots$. It is known (Zygmund [**3**, Vol. 2, p. 150]) that $\alpha=(\alpha_k)$ is positive if and only if there exists an increasing function F_α, defined and bounded on $[0, 2\pi]$ with

$$\alpha_k = \frac{1}{\pi}\int_0^{2\pi}\cos k\theta\, dF_\alpha(\theta), \quad 0 = \frac{1}{\pi}\int_0^{2\pi}\sin k\theta\, dF_\alpha(\theta), \quad k = 0, 1, \cdots .$$

It is customary to extend F_α to all of the reals by $F_\alpha(\theta+2\pi)=F_\alpha(\theta)+F_\alpha(2\pi)-F_\alpha(0)$. Let us say F_α generates α. When α and β are positive, the sequences $c\alpha=(c\alpha_k)$ $(c\geqq 0)$, $\alpha+\beta=(\alpha_k+\beta_k)$, $\alpha * \beta=(\alpha_k\beta_k)$ are again positive. In the latter case, $\alpha * \beta$ is generated by

$$F_\gamma(\theta) = \frac{1}{\pi}\int_0^{2\pi} F_\alpha(\theta - t)\, dF_\beta(t), \qquad \gamma = \alpha * \beta.$$

(See Zygmund [**3**, Vol. 1, p. 38].) For example, if $(a_k)_{k\geqq n}$ and $(b_k)_{k\geqq n}$ are positive and $g(z)$ $\sum_{k=n}^\infty a_kz^k$, $h(z)=\sum_{k=n}^\infty b_kz^k$, we have

$$E_{n-1}(H_1 * g * h, R) = a_nb_nR^n, \qquad 0 < R \leqq 1.$$

Simple conditions that α be positive are

$$(9)\quad \alpha_k \geqq 0,\ \Delta\alpha_k = \alpha_{k+1} - \alpha_k \leqq 0,\ \Delta^2\alpha_k = \Delta(\Delta\alpha_k) \geqq 0,\ k = 0, 1, \cdots .$$

To see this, one applies partial summation twice to the series in (4) to obtain the inequality (4). Note that if α and β satisfy (9) then so do $c\alpha$ $(c\geqq 0)$, $\alpha+\beta$, $\alpha * \beta$, and $(\alpha_{k+p})_{k\geqq 0}$ $(p\geqq 0)$. Examples of sequences satisfying (9) are moment sequences $\alpha_k=\int_0^1 t^k dq(t)$ with q increasing and bounded in $[0, 1]$. We have immediately

PROPOSITION 1. *If* $g(z)=\sum_{k=p}^\infty a_kz^k$ *and* $(a_{k+p})_{k\geqq 0}$ *satisfies* (9), *then*

$$(10)\qquad E_{n-1}(H_1 * g, R) = a_nR^n, \qquad 0 < R \leqq 1,\ n \geqq p.$$

Now we can state an extension of Babenko's theorem which is contained in (10). If p is a nonnegative real number we may define the pth derivative of $f(z)=\sum_{k=0}^\infty c_kz^k$ $(f\in H)$ by the equation

$$(11)\qquad z^pf^{(p)}(z) = \sum_{k=0}^\infty \frac{\Gamma(k+1)}{\Gamma(k-p+1)}\, c_kz^k$$

where z^p is real for real z and $1/\Gamma(-j)=0$ if $j=0, 1, \cdots$. Let

$B^{(p)}=\{f\in H: |f^{(p)}(z)|\leqq 1 \text{ for } |z|<1\}$. Then we have the

COROLLARY. *For each* $p\geqq 0$ *and* $n\geqq [p]$,

$$(12)\qquad E_{n-1}(B^{(p)},R)=\frac{\Gamma(n-p+1)}{\Gamma(k+1)}R^n,\qquad 0<R\leqq 1.$$

PROOF. We have $f\in B^{(p)}$ if and only if $f(z)=\sum_{k=0}^{[p]-1} c_k z^k+h*g(z)$ with $h(z)=z^p f^{(p)}(z)\in H_1$ and $g(z)=\sum_{k=[p]}^{\infty} a_k z^k$, where $a_k=[\Gamma(k-p+1)\cdot\Gamma(k+1)^{-1}]$. One verifies immediately that the sequence $(a_k)_{k\geqq p}$ satisfies (9).

We can give another interpretation of our main result by means of differential operators. Let T be the operator uD, that is,

$$(13)\qquad Tf(z)=zf'(z)=\sum_{k=0}^{\infty} kc_k z^k \quad \text{if } f(z)=\sum_{k=0}^{\infty} c_k z^k$$

If $\psi(w)$ is a function defined for $w=n, n+1,\cdots$, where n is a nonnegative integer, and if

$$(14)\qquad \limsup_{k\to\infty} |\psi(k)|^{1/k}\leqq 1,$$

we define the differential operator $\psi(T)$ from H into H by

$$(15)\qquad \psi(T)f(z)=\sum_{k=n}^{\infty}\psi(k)c_k z^k \quad \text{if } f(z)=\sum_{k=0}^{\infty} c_k z^k.$$

That this definition is reasonable can be seen from the following example. Let $\psi(w)$ be an entire function of order not exceeding one and of minimal type if the order is one. Then $\limsup_{k\to\infty}(\sum_{j=0}^{\infty}|d_j|k^j)^{1/k}\leqq 1$, where $\psi(w)=\sum_{j=0}^{\infty} d_j w^j$. (See Hille [**4**, Vol. 2, p. 183].) It follows that

$$\psi(T)f(z)=\sum_{k=0}^{\infty}\psi(k)c_k z^k=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} d_j k^j c_k z^k=\sum_{j=0}^{\infty} d_j T^j f(z),$$

the change in the order of summation being possible by the absolute convergence of the double series. The series on the right is the definition of $\psi(T)f$ to be found in Hille [**4**, Vol. 2, p. 48]. Let $S_\psi=\{f\in H: \psi(T)f\in H_1\}$ where $\psi(T)f$ is defined by (15). Now we can state

PROPOSITION 2. *Let* $\psi(w)$ *be defined and nonzero for* $w=n, n+1,\cdots$ *and satisfy* (14). *Then if* $(1/\psi(k))_{k\geqq n}$ *is a positive sequence, we have*

$$(16)\qquad E_{n-1}(S_\psi,R)=R^n/\psi(n),\qquad 0<R\leqq 1.$$

PROOF. We have $S=H_1 * g$, where $g(z)=\sum_{k=n}^{\infty}\psi(k)^{-1}z^k$, and (16) follows from (6).

As a special case of Proposition 2 we state

PROPOSITION 3. *Formula* (16) *holds if* $\psi(x)$ *is defined for* $x\geqq n$, *satisfies* (14) *and also*

$$\psi(x)>0,\ \psi'(x)\geqq 0,\ 2\psi'(x)^2\geqq\psi(x)\psi''(x),\qquad x\geqq n. \tag{17}$$

PROOF. Condition (17) means that $1/\psi(x)$ is positive, increasing, and convex in $x\geqq n$. Hence $a_k=1/\psi(k)$, $k\geqq n$, satisfies (9).

For example, if $\psi(x)=x^p$ ($p\geqq 0$) or $\psi(x)=\prod_{k=1}^{\infty}(1+x/r_k)$ where $r_k>0$ and $\sum_{k=1}^{\infty}r_k^{-1}<+\infty$, then $\psi(x)$ satisfies the conditions of Proposition 3 (see Hille [4, Vol. 2, p. 195]). So does $\psi(x)=\sum_{j=0}^{m}d_jx^j$, if the d_j are real and $d_m>0$, for $x\geqq n$, when n is sufficiently large.

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