THE INTEGRAL OF AN INVARIANT UNIMODAL FUNCTION OVER AN INVARIANT CONVEX SET —AN INEQUALITY AND APPLICATIONS[1]

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1. **Introduction and summary.** It is well known that the integral $\int_{-a+\theta}^{a+\theta} f(x)dx$, $a>0$, of a nonnegative function $f(x)$ on the real line, which is unimodal, i.e., $f(kx) \geqq f(x)$, $0 \leqq k \leqq 1$, and symmetrical about the origin, is a monotonically decreasing function of $|\theta|$. An immediate probabilistic consequence of this is the fact that, if a random variable X has a unimodal probability density function symmetric about the origin, and Y is any independently distributed random variable, then $\Pr\{|X| \geqq a\} \leqq \Pr\{|X+Y| \geqq a\}$ for any real a. T. W. Anderson [1] has extended the aforementioned monotonicity property to integrals of functions on a Euclidean n-space $\mathfrak{L}_n$ by replacing the symmetric interval of the real line by a convex set of $\mathfrak{L}_n$ symmetric about the origin, and formulating the following definition of unimodality of functions on $\mathfrak{L}_n$.

DEFINITION 1. *A function $f(x)$ on $\mathfrak{L}_n$ is said to be unimodal if the set $K_u = \{x | f(x) \geqq u\}$ is convex for each $u \geqq 0$.*

More specifically, he has proved the following:

THEOREM 1. *Let E be a convex set in $\mathfrak{L}_n$, symmetric about the origin. Let $f(x) > 0$ be a function such that* (i) $f(x) = f(-x)$, (ii) $\{x | f(x) > u\} = K_u$ *is convex for every u,* $(0 < u < \infty)$, *and* (iii) $\int_E f(x)dx < \infty$ (*in the Lebesque sense*). *Then*

$$\int_E f(x + ky)\, dx \geqq \int_E f(x + y)\, dx \tag{1}$$

for $0 \leqq k \leqq 1$.

Anderson has also discussed some analogues of the probability inequality mentioned above and many other probabilistic and statistical applications.

In §2 we have obtained a generalization of the Theorem 1 by relaxing the condition of symmetry about the origin, on the function f and the set E, to a restriction of invariance with respect to finite groups of linear transformations of $\mathfrak{L}_n$, and we have indicated the analogues of some of the probability inequalities in [1].

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In §3 we have discussed some particular cases by considering invariance with respect to the group of reflections in the origin, the permutation group, and the cyclic permutation group in n-space. It has been shown that the integral of a symmetric unimodal function over a symmetric convex region, is an S-concave (in Ostrowski's sense) function of the translation parameter.

In the final §4, we have stated and have outlined a somewhat different proof of a slightly different version of the inequality without assuming the group of transformations to be finite.

2. **The inequality with invariance w.r.t. a finite group G.** Let $G=\{g_i, i=1, 2, \cdots, N\}$ be a finite group of Lebesgue measure-preserving linear transformations of $\mathfrak{L}_n$ onto $\mathfrak{L}_n$. Let E be a convex set of n-space, invariant under G, or G-invariant, i.e. $x\in E$ implies $g_ix\in E$, $i=1, 2, \cdots, N$. Let $f(x)\geqq 0$ be a function on n-space satisfying

(2)
- (i) the unimodality condition: $\{x \mid f(x)\geqq u\}=K_u$ is convex for every u, $0<u<\infty$,
- (ii) G-invariance condition: $f(g_ix)=f(x)$, $i=1, 2, \cdots, N$, for each x in $\mathfrak{L}_n$, and
- (iii) $\int_E f(x)dx<\infty$ in the Lebesgue sense.

For a set $\alpha=\{\alpha_1, \alpha_2, \cdots, \alpha_N\}$, $\alpha_i\geqq 0$, $i=1, 2, \cdots, N$, $\sum_{i=1}^N \alpha_i=1$, and a vector y of n-space let us define

$$\alpha(y) = \sum_{i=1}^{N} \alpha_i g_i y. \tag{3}$$

Then we have, as a generalization of the Theorem 1, the following:

THEOREM 2. *For each set $\alpha=\{\alpha_1, \cdots, \alpha_N\}$, $\alpha_i\geqq 0$, $\sum\alpha_i=1$, and vector y of $\mathfrak{L}_n$ we have*

$$\int_E f(x+\alpha(y))\,dx \geqq \int_E f(x+y)\,dx, \tag{4}$$

where $f\geqq 0$ and E are G-invariant, f is unimodal, E is convex and $\alpha(y)$ is defined by (3), *provided $\int_E f(x)dx<\infty$.*

PROOF. We have to show, equivalently, that

$$\int_{E+\alpha(y)} f(x)\,dx \geqq \int_{E+y} f(x)\,dx, \tag{5}$$

where $E+y$ is the set E translated by the vector y. Now it is easy to verify that, because of the convexity of E and K_u

$$(6)\qquad \{E+\alpha(y)\}\cap K_u \supset \sum_{i=1}^{N} \alpha_i[\{E+g_iy\}\cap K_u],$$

where the summation symbol $\sum$ on the right hand side of the inclusion relation corresponds to the sum, called the Minkowski sum [4], defined by

$$(7)\qquad A+B=\{a+b \mid a\in A,\, b\in B\}$$

for any two sets A, B of $\mathfrak{L}_n$, and where multiple cA of a set A of $\mathfrak{L}_n$ by a real c is defined by $cA=\{ca \mid a\in A\}$. Let $\mu(\cdot)$ be the Lebesgue measure of sets in $\mathfrak{L}_n$. Then we have by the Brunn-Minkowski theorem [2], [3], [4],

$$(8)\qquad \mu^{1/n}\left(\sum_{i=1}^{N} \alpha_i[\{E+g_iy\}\cap K_u]\right) \geqq \sum_{i=1}^{N} \alpha_i\mu^{1/n}(\{E+g_iy\}\cap K_u).$$

But because of the invariance of f and E and linearity and measure preserving properties of the transformations g_i, $i=1, 2, \cdots, N$, we have

$$(9)\qquad \mu(\{E+g_iy\}\cap K_u) = \mu(\{E+y\}\cap K_u).$$

Combining (6), (8) and (9) we get

$$(10)\qquad H(u) = \mu(\{E+\alpha(y)\}\cap K_u) \geqq \mu(\{E+y\}\cap K_u) = H^*(u).$$

Because of the definition of Lebesgue-Stieltjes integrals we can write

$$(11)\qquad \int_{E+\alpha(y)} f(x)\,dx - \int_{E+y} f(x)\,dx = \int_0^\infty u\,d[H^*(u)-H(u)].$$

The right hand side of (11) is nonnegative, which may be verified by using integration by parts as in the proof of the Theorem 1 of [1]. This completes the proof of the Theorem 2.

As in [1], it may be noted that we obtain strict inequality in (4) if, and only if, for at least one u, $H(u)>H^*(u)$, since $H(u)$ is continuous on the left. For $H(u)=H^*(u)$ we need equality in (8), which is a consequence of the Brünn-Minkowski theorem. The condition for equality may, therefore, be stated as:

COROLLARY 1. *In the Theorem* 2, *the equality in* (4) *holds if, and only if,* $(E+g_iy)\cap K_u$ *are similarly oriented for each* u.

COROLLARY 2. *If the probability density function* $f(x)$ *of a random* n-*vector* X *satisfies the conditions* (2) *and* E *is a convex set of* n-*space invariant under* G, *then for any* n-*vector* y *and set* α, $\Pr\{X+\alpha(y)\in E\}$

$\geqq \Pr\{X+y\in E\}$. *Furthermore, if* $h(x)$ *is a G-invariant function such that* $\{x \mid h(x)\leqq v\}$ *is convex, then* $\Pr\{h(X+\alpha(y))\leqq v\}\geqq \Pr\{h(X+y)\leqq v\}$.

The proof of the following corollary is analogous to the proof of the Theorem 2 of [**1**].

COROLLARY 3. *Let the probability density function* $f(x)$ *of a random n-vector* X *satisfy the conditions* (2) *and let* Y *be any independently distributed random n-vector. Then for any set* $\alpha=\{\alpha_1, \cdots, \alpha_N\}$, $\alpha_1\geqq 0$, $\sum \alpha_i=1$, *and any convex G-invariant set E of n-space*

$$\Pr\{X+\alpha(Y)\in E\} \geqq \Pr\{X+Y\in E\}. \tag{12}$$

Furthermore, if $h(x)$ *is a G-invariant function such that* $\{x \mid h(x)\leqq v\}$ *is convex, then*

$$\Pr\{h(X+\alpha(Y))\leqq v\} \geqq \Pr\{h(X+Y)\leqq v\}. \tag{13}$$

3. **Some particular cases.** (i) If the group G in §2 is the group of reflections in the origin the Theorem 2 reduces to the above stated Theorem 1 of Anderson [**1**].

(ii) An important particular case of the Theorem 2 is obtained if the group G is the permutation group in $\mathfrak{L}_n$.

DEFINITION 2 (OSTROWSKI [**6**]). *A function* $G(y)$ *on* $\mathfrak{L}_n$ *is said to be S-concave if, for each doubly stochastic matrix S, of order n, and each y in* $\mathfrak{L}_n$

$$G(Sy) \geqq G(y). \tag{14}$$

Now Birkhoff's theorem [**5**] states that the set of doubly stochastic matrices of order n is a convex polyhedron with $N=n!$ permutation matrices P_i, $i=1, 2, \cdots, N$, as the vertices. Thus every doubly stochastic matrix $S=\sum_{i=1}^{N}\alpha_i P_i$ for some set $\alpha=\{\alpha_1, \cdots, \alpha_N\}$, $\alpha_i\geqq 0$, $\sum_{i=1}^{n}\alpha_1=1$. Hence we have the following Theorem 3 as a particular case of the Theorem 2.

THEOREM 3. *Let a function* $f(x)\geqq 0$ *on* $\mathfrak{L}_n$ *be symmetric (w.r.t. permutations) and satisfy the conditions* (i) *and* (iii) *of* (2). *Let E be a convex, symmetric (w.r.t. permutations) set of* $\mathfrak{L}_n$. *Then* $\int_E f(x+y)dx$ *is an S-concave function of y, i.e.,*

$$\int_E f(x+Sy)\,dx \geqq \int_E f(x+y)\,dx, \tag{15}$$

for any doubly stochastic matrix S.

It is also well known [6] that, given two n-vectors y and z, there exists a doubly stochastic matrix S, such that $z=Sy$ if, and only if,

$$\begin{aligned} &z_{(1)} + \cdots + z_{(k)} \leqq y_{(1)} + \cdots + y_{(k)}, \qquad k = 1, 2, \cdots, n-1, \\ &z_{(1)} + \cdots + z_{(n)} = y_{(1)} + \cdots + y_{(n)}, \end{aligned} \tag{16}$$

where $y_{(i)}$ and $z_{(i)}$, $i=1, 2, \cdots, n$ are the coordinates of y and z, ordered in nonincreasing order of magnitude. The conclusion (15) of the Theorem 3 may thus be expressed as: For any two vectors y and z of n-space, satisfying (16) we have

$$\int_E f(x+z)\,dx \geqq \int_E f(x+y)\,dx. \tag{17}$$

Corollary 4. *If the probability density function $f(x)$ of a random n-vector X is unimodal and symmetric w.r.t. permutations of the coordinates of x, and Y is an independently distributed random vector then for any convex symmetric set E of $\mathfrak{L}_n$, and any doubly-stochastic matrix S, we have*

$$\Pr\{X + SY \in E\} \geqq \Pr\{X + Y \in E\}. \tag{18}$$

Furthermore, if h is a symmetric function such that $\{x \mid h(x) \leqq v\}$ is convex then

$$\Pr\{h(X + SY) \leqq v\} \geqq \Pr\{h(X + Y) \leqq v\}. \tag{19}$$

(iii) Now let the transformation group of §2 be the group of the cyclic permutations of n coordinates. One has the matrix representation for this group as $g_i = p^{i-1}$, $i=1, 2, \cdots, n$, where P is a permutation matrix given by

$$P = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Thus for any vector y of $\mathfrak{L}_n$ and a set $\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, $\alpha_i \geqq 0$, $i=1, 2, \cdots, n$, $\sum \alpha_i = 1$ we have

$$\alpha(y) = \sum_{i=1}^n \alpha_i g_i y = \sum_{i=1}^n \alpha_i P^{i+1} y = C(\alpha) y,$$

where $C(\alpha)$ is the doubly stochastic circulant matrix given by $C(\alpha) = (c_{ij})$, $c_{ij} = \alpha_k$, $k = k(i, j) = i+j-1 \bmod (n)$ or more explicitly,

$$C(\alpha) = \begin{pmatrix} \alpha_1 & \alpha_2 \cdots \alpha_{n-1} & \alpha_n \\ \alpha_2 & \alpha_3 \cdots \alpha_n & \alpha_1 \\ \alpha_3 & \alpha_4 \cdots \alpha_1 & \alpha_2 \\ \cdot & \cdots\cdots & \cdot \\ \cdot & \cdots\cdots & \cdot \\ \alpha_n & \alpha_1 \cdots \alpha_{n-2} & \alpha_{n-1} \end{pmatrix}.$$

Also, it is easy to verify that the set of all doubly stochastic circulant matrices form a convex polyhedron with the permutation matrices p^{i-1}, $i=1, 2, \cdots, n$, as the vertices. We have, therefore, the following Theorem 4 as another particular case of the Theorem 2.

THEOREM 4. *Let a function $f(x) \geqq 0$ on n-space by unimodal and invariant under cyclic permutations. Let E be a convex set of n-space symmetric with respect to cyclic permutations. Let $\int_E f(x)dx < \infty$. Then for any doubly stochastic circulant matrix C and any vector y of n-space we have*

$$\int_E f(x + Cy)\,dx \geqq \int_E f(x + y)\,dx. \tag{18}$$

It is easy to write down analogues of the Corollary 4 for this case. One may similarly write down, with ease, the particular case of the Theorem 2 when the function $f(x)$ and the set E are invariant under the group of 2^n reflections in the coordinate planes.

4. **The inequality.** We shall now outline a somewhat different proof of a somewhat different and generalized version of the inequality of §2 without the finiteness condition on the group G.

THEOREM 5. *Let $G=\{g\}$ be a group of linear Lebesgue measure-preserving transformations of $\mathfrak{L}_n$ onto $\mathfrak{L}_n$. Let E be a convex, G-invariant region of $\mathfrak{L}_n$. Let f be a nonnegative real-valued, G-invariant and unimodal function on $\mathfrak{L}_n$. Then for arbitrary y in $\mathfrak{L}_n$ we have*

$$\int_E f(x + z)\,dx \geqq \int_E f(x + y)\,dx, \tag{19}$$

where z is any point in the convex-hull of the G-orbit of y.

PROOF. The theorem can be proved along the lines of the proof of the Theorem 2, by using the generalized version of the Brünn-Minkowski theorem due to Dinghas [2], [3], [4]. However, the argument may be simplified as follows, by using a twist suggested by

Kemperman in a personal communication.

The crucial step in the proof of the Theorem 2 is the statement (10), which holds without requiring G to be finite. To see this let us fix y and write,

$$Z = \{z \mid \phi(z) \geqq \phi(y)\}, \tag{20}$$

where

$$\phi(z) = \mu^{1/n}((E + z) \cap K_u). \tag{21}$$

Then by the Brünn-Minkowski theorem it follows that,

$$\phi(\lambda z_1 + (1 - \lambda)\, z_2) \geqq \lambda\phi(z_1) + (1 - \lambda)\phi(z_2), \quad \text{for } 0 \leqq \lambda_1 \leqq 1. \tag{22}$$

Hence for any y in $\mathfrak{L}_n$ the set Z of (20) is convex. Furthermore as in the proof of the Theorem 2 it can be verified that $gy \in Z$ for each $g \in G$. Therefore, for any point z in the convex-hull of $\{gy \mid g \in G\}$ we have

$$\mu((E + z) \cap K_u) \geqq \mu((E + y) \cap K_u), \tag{23}$$

which is analogous to the statement (10). The proof from here on is the same as the proof of the Theorem 2.

The analogue of the Corollary 2 for the Theorem 5 is easy to formulate. A group G of special interest in probability and statistics is the group of orthogonal transformations. The G-orbit of any y for this group is the sphere $\sum_{i=1}^{n} x_i^2 = \|y\|^2$ in $\mathfrak{L}_n$. This special case may be studied as in §3 without any difficulty.

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