# A NOTE ON $IC$-$p$ GROUPS

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In [1], Bauman defines an $IC$ group as a finite group in which every intersection of two subgroups neither of which contains the other has every Sylow subgroup cyclic. His Theorem 1 shows that an $IC$ group has a normal 2-complement unless an $S_2$ is abelian or quaternion. Using some of his methods we prove a more general result.

DEFINITION. A finite group $G$ is an $IC$-$p$ group if whenever $H$, $K \subseteq G$, $H \not\subseteq K$, $K \not\subseteq H$ and $H \cap K$ is a $p$-group, then $H \cap K$ is cyclic.

Our main result is

THEOREM. *Let $G$ be an $IC$-$p$ group which is not a $p$-group. Then an $S_p$ of $G$ is either*

(i) *cyclic,*

(ii) *of period $p$ and order $\leqq p^3$ or*

(iii) *quaternion.*

We begin by observing that if $G$ is an $IC$-$p$ group then so is every subgroup and if $P \triangle G$ is a $p$-group then $G/P$ is an $IC$-$p$ group. Unlike the situation with $IC$ groups, it is false that all quotients of $IC$-$p$ groups also have the property. An example is the direct product of the quaternion group with an elementary abelian group of order $p^2$ for any odd prime $p$. This is an $IC$-$p$ group but its quotient by the subgroup of order 2 is not.

LEMMA 1. *Let $G$ be a $p$-group which is an $IC$-$p$ group and let $P \triangle G$ be a noncyclic subgroup. Then $G/P$ is cyclic.*

PROOF. If $G/P$ has more than one maximal subgroup then each corresponds to a maximal subgroup of $G$ and the intersection of two of these subgroups is cyclic and contains $P$. This cannot happen and thus $G/P$ has a unique maximal subgroup and hence is cyclic.

LEMMA 2. *An $IC$-$p$ group is $p$-normal.*

PROOF. This is essentially Lemma 2 of Bauman's paper and his proof works here.

LEMMA 3. *An $IC$-$p$ group of period $p$ has order $\leqq p^3$.*

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PROOF. The group in question, $P$, is a $p$-group and if its order were $\geqq p^3$ it would have a normal subgroup $A$ of order $p^2$. Since $A$ is not cyclic, $P/A$ is cyclic and has period and hence order $p$ by Lemma 1. Therefore $|P| = p^3$.

LEMMA 4. *If $P$ is a 2-group and $Z = \mathfrak{Z}(P)$ then $P/Z$ is not the quaternion group. If $P/Z$ is abelian and $P$ is generalized quaternion, then $P$ is quaternion.*

PROOF. If $P/Z$ is quaternion there are distinct $A$, $B \supseteq Z$ of index 2 in $P$ with $A/Z$ and $B/Z$ cyclic. Thus $A$ and $B$ are abelian and $A \cap B \subseteq \mathfrak{Z}(\langle A, B\rangle) = \mathfrak{Z}(P) = Z$. Therefore $[P:Z] = 4$ contradicting $P/Z$ quaternion. The second statement follows from an examination of the generalized quaternion groups.

LEMMA 5. *Let $G$ have a cyclic normal $p$-subgroup $P$. If $u$ is any element of $G$ of order prime to $p$ which centralizes some element of $P$ then $u$ centralizes all of $P$.*

PROOF. Let $P = \langle x\rangle$ and suppose $u^{-1}xu = x^i$, $u^{-1}x^r u = x^r$ where $r < p^k = |P|$. Since $x^r = u^{-1}x^r u = x^{ri}$, we have $p^k | (ri - r)$ and since $p^k > r$, $p | (i-1)$ and $i \equiv 1 \bmod p$. If the order of $u$ is $s$ then $x = u^{-s}xu^s = x^{i^s}$ and $p^k | (i^s - 1)$. Since $1 + i + i^2 + \cdots + i^{s-1} \equiv 1 + 1 + \cdots + 1 = s \not\equiv 0 \bmod p$, we have $p^k | (i-1)$ and $x^i = x$. Therefore $u$ centralizes $P$.

PROOF OF THE THEOREM. If $H \subseteq G$ is not a $p$-group and $p | [G:H]$, let $P_0$ be an $S_p$ of $H$ and let $P \supseteq P_0$ be one of $G$. Then $P_0 = P \cap H$ and $P \not\subseteq H$ and $H \not\subseteq P$ and thus $P_0$ is cyclic. In particular, if any $P_0 < P$ is normalized by a $p'$ element $u$ of $G$, then by letting $H = \langle P_0, u\rangle$ we see that $P_0$ is cyclic. Similarly, if $N$ is a normal $p'$ subgroup of $G$ and $P_0 < P$ is arbitrary, then letting $H = P_0 N$ we conclude that $P_0$ is cyclic. In this situation let $x \in \mathfrak{Z}(P)$ have order $p$. If $y \in P$ has order $p$ then letting $P_0 = \langle x, y\rangle$ we have either $P_0 = P$ and $|P| \leq p^2$ or $P_0$ is cyclic. Therefore if $|P| > p^2$, $\langle x\rangle$ contains all elements of order $p$ and $P$ is either cyclic or generalized quaternion. Since a generalized quaternion group contains the noncyclic quaternion group, the inclusion cannot be proper. We have therefore proved the theorem in the case where there is some normal $p'$ subgroup.

Suppose that in the general case the theorem is false. Let $G$ be a counterexample of minimal order with $S_p$ $P$. Since the $IC$-$p$ property is inherited by subgroups, $P$ is maximal in $G$. If $Z = \mathfrak{Z}(P)$ then $P \subseteq \mathfrak{N}(Z)$ and either $Z \triangle G$ or $\mathfrak{N}(Z) = P$. We may apply Grun's Theorem by Lemma 2 and thus in the latter case $G$ has a normal $p$-complement. Since $G$ can have no normal $p'$ subgroup $G$ must be a $p$-group which we are assuming is not the case. Therefore $Z \triangle G$ and hence $P\subseteq\mathfrak{C}(Z)\Delta G$. If $P$ is not normal in $G$ then $\mathfrak{C}(Z)=G$ and $Z\subseteq\mathfrak{Z}(G)$.

Continuing with the assumption that $P$ is not normal, suppose $P\supseteq Z_0\supseteq Z$, $Z_0\Delta G$ and $P/Z_0$ is abelian. Since $P/Z_0$ is its own normalizer in $G/Z_0$ we may apply Burnside's Theorem to conclude that $G/Z_0$ has a normal $p$-complement. In particular, if $P/Z$ is abelian, $G/Z$ has a normal $p$-complement which has the central $S_p\,Z$ and thus $G$ has a normal $p$-complement which is a contradiction.

By the theorem applied to $G/Z$ and by Lemma 4 we may conclude that $p$ is odd and $P/Z$ is the nonabelian group of period $p$ and order $p^3$. We let $Z_1$ be the inverse image of the center of $P/Z$. We may conclude as before from Grun's Theorem that either $Z_1\Delta G$ or else $G/Z$ has a normal $p$-complement which we know is impossible since $Z\subseteq\mathfrak{Z}(G)$. Since $P/Z_1$ is abelian, we may let $Z_0=Z_1$ and conclude that $G/Z_1$ has a normal $p$-complement $M$. Since $Z_1<P$ it is cyclic and since every $p'$ element of $M$ centralizes $Z\subseteq Z_1$, by Lemma 5 it also centralizes $Z_1$ which therefore is a central $S_p$ of $M$. Hence $M$ has a normal $p$-complement and this yields a contradiction. We therefore must have $P\Delta G$.

Suppose now that $P/Z$ is abelian. If $p=2$ then by Lemma 4 $P$ is not generalized quaternion and thus has more than one involution. If $P$ is not abelian then $Z<P$ is cyclic and $P$ has a noncentral involution $x$. Since we are assuming $P$ is not abelian, $P/Z$ is not cyclic and thus by the theorem applied to $G/Z$ it is elementary abelian and $\langle Z, x\rangle\Delta P$. Since $\langle Z, x\rangle$ is not cyclic, by Lemma 1 its quotient in $P$ is cyclic and thus of order 2 and $[P\colon Z]=4$. Let $Z_0$ be of index 2 in $Z$. If $Z_0>1$ then by induction, since $Z_0\Delta G$, either $P/Z_0$ is elementary abelian or quaternion. Since $Z/Z_0$ does not contain the image of $x$, $P/Z_0$ has more than one involution and thus $P/Z_0$ is elementary abelian and the quotient $P/\langle Z_0, x\rangle$ is not cyclic. This contradiction shows that $Z_0=1$ and thus $|P|=8$ and therefore $P$ is dihedral and has exactly two elementary abelian maximal subgroups. Each is therefore normalized by any element of $G$ of odd order and thus by our earlier remarks must be cyclic. This contradiction shows that if $P/Z$ is abelian, either $p$ is odd or $P$ is abelian and since the class of $P$ is $\leqq 2$ it is a regular $p$-group in either case. Therefore the subgroup $P_1$ generated by the elements of order $p$ has period $p$. $P_1\neq P$ by Lemma 3 and thus $P_1$, being normal in $G$, is cyclic and there is a unique subgroup of order $p$. Therefore $P$ is cyclic and we have a contradiction.

Since $P/Z$ is not abelian we may conclude as before by applying the theorem to $G/Z$ that $p$ is odd and $P/Z$ is the nonabelian group of period $p$ and order $p^3$. Let $Z_1 \triangle G$ be the inverse image in $P$ of $\mathfrak{Z}(P/Z)$. Now $Z_1$ is cyclic but no maximal subgroup of $P$ is cyclic for this would correspond to an element of order $p^2$ in $P/Z$. Therefore no maximal subgroup of $P$ is normalized by any $p'$ element of $G$.

Now let $y \in G$ have prime order $q \neq p$. Since $y$ permutes the $p+1$ maximal subgroups of $P$ containing $Z_1$ and fixes none of them, $q \mid p+1$. Since $\mathfrak{C}_P(Z_1) < P$ and is normal in $G$ it must equal $Z_1$. Therefore $\mathfrak{C}_G(Z_1)$ has the central $S_p$ $Z_1$ and thus has a normal $p$-complement which as we have seen must be trivial. Hence $y$ does not centralize $Z_1$ and thus by Lemma 5 $y$ acts without nontrivial fixed points on $Z_1$ and $q \mid p-1$. Thus $q=2$ and $y$ is an involution. Now $y$ fixes no $x \in P - Z_1$ or else it normalizes $\langle Z_1, x\rangle$. Therefore $y$ is an involution acting without nontrivial fixed points on $P$ which must therefore be abelian. This contradiction proves the theorem.

As an application of the theorem we prove the following.

COROLLARY. *If $G$ is a $p$-solvable IC-$p$ group, then the $p$-length of $G$ is* 1.

PROOF. We must show that $\mathfrak{O}_{p'pp'}(G) = G$. Assume to the contrary that $\mathfrak{O}_{p'pp'}(G)$ does not contain an $S_p$ $P$ of $G$. By Lemma 1.2.3 of [2], $\mathfrak{O}_{p'p}(G)/\mathfrak{O}_{p'}(G)$ contains its own centralizer in $G/\mathfrak{O}_{p'}(G)$. Therefore $\mathfrak{O}_{p'p}(G)/\mathfrak{O}_{p'}(G)$ is not central in $P\mathfrak{O}_{p'}(G)/\mathfrak{O}_{p'}(G)$ and in particular $P$ is not abelian and since a normal subgroup of order $p$ in a $p$-group is always central, $p^2 \mid [\mathfrak{O}_{p'p}(G):\mathfrak{O}_{p'}(G)]$. By the theorem we may conclude that $P$ either has period $p$ or is quaternion. Since certainly $\mathfrak{O}_{p'pp'}(G) \not\supseteq P$ we know that $P \cap \mathfrak{O}_{p'pp'}(G)$ is cyclic and thus $\mathfrak{O}_{p'p}(G)/\mathfrak{O}_{p'}(G)$ is cyclic. Since its order is $\geqq p^2$ it cannot have period $p$ and thus $p=2$ and $P$ is quaternion. Therefore $\mathfrak{O}_{p'p}(G)/\mathfrak{O}_{p'}(G)$ is a cyclic normal subgroup of order 4 of $G/\mathfrak{O}_{p'}(G)$ which is thus centralized by every element of odd order in $G/\mathfrak{O}_{p'}(G)$. We conclude that $G/\mathfrak{O}_{p'}(G)$ is a 2-group and this contradiction completes the proof.

## REFERENCES

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