QUOTIENT UNIFORMITIES[1]

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1. **Introduction.** Let f be a function from a set X onto a set Y, let $\mathcal{U}$ be a uniformity for X, and let $\boldsymbol{f}$: $X\times X\to Y\times Y$ be defined by $\boldsymbol{f}(x, y)=(f(x), f(y))$ for $(x, y)\in X\times X$. In this note we determine necessary and sufficient conditions that $\boldsymbol{f}(\mathcal{U})$ be a uniformity, and point out that, if $\boldsymbol{f}(\mathcal{U})$ is a uniformity, it is the quotient uniformity for Y relative to f. Then we obtain a theorem relating the quotient topology and the quotient uniformity on Y relative to f. In [2] these results will be applied to determine necessary and sufficient conditions that a decomposition of a pseudo-metrizable space be pseudo-metrizable.

Throughout the paper, f, X, Y will be as above. If $x\in X$, let $R[x]$ be the set of all $z\in X$ such that $f(z)=f(x)$. Otherwise, whatever notation or terminology is not defined in this paper is as in [3]. In particular, we do not require that uniformities be separated.

2. **Quotient uniformities.** If X and Y are topological spaces and f is continuous, we call f a map. If Y has the quotient topology relative to f, we call f a quotient map. Equivalently, the terms identification topology and quasi-compact map are often used. The quotient topology on Y is the largest topology which makes f continuous; it is the unique topology on Y such that for each topological space Z and each function g: $Y\to Z$, g is continuous if and only if $g\circ f$ is continuous. Accordingly, given a uniformity $\mathcal{U}$ for X, we are inclined to use one or the other of the following conditions as the definition of the quotient uniformity $\mathcal{V}$ for Y relative to f.

(A) $\mathcal{V}$ is the largest uniformity for Y which makes f uniformly continuous.

(B) $\mathcal{V}$ is a uniformity for Y such that for each uniform space Z and each function g: $Y\to Z$, g is uniformly continuous if and only if $g\circ f$ is uniformly continuous.

It is easy to see that in each of (A), (B), $\mathcal{V}$ is unique, if it exists. There is, in fact, a $\mathcal{V}$ which satisfies both of (A) and (B). For let $\mathcal{V}_1$ be the family of all sets $V\subset Y\times Y$ such that V contains the diagonal of $Y\times Y$ and $\boldsymbol{f}^{-1}[V]\in\mathcal{U}$. Then let $\mathcal{V}$ consist of all those sets V_0 con-

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tained in $Y\times Y$ for which there exist sets V_n satisfying $V_n\in\mathcal{V}_1$ and $V_n\circ V_n\subset V_{n-1}$ for $n\geqq 1$. It is easy to verify that $\mathcal{V}$ is the largest uniformity making f uniformly continuous, and hence that $\mathcal{V}$ satisfies (A). Now give Y the uniformity $\mathcal{V}$ and consider $g\colon Y\to Z$ where Z has uniformity $\mathcal{W}$. If g is uniformly continuous, then, trivially, so is $g\circ f$. Conversely, if $g\circ f$ is uniformly continuous, consider $W_0\in\mathcal{W}$; there exist $W_n\in\mathcal{W}$ such that $W_n\circ W_n\subset W_{n-1}$ for $n\geqq 1$. Then, since $f^{-1}\circ g^{-1}[W_n]\in\mathcal{U}$, we have $g^{-1}[W_n]\in\mathcal{V}_1$ for each $n\geqq 1$; and, since $g^{-1}[W_n]\circ g^{-1}[W_n]\subset g^{-1}[W_{n-1}]$ for each $n\geqq 1$, it follows that $g^{-1}[W_0]\in\mathcal{V}$ and that g is uniformly continuous. We call the unique $\mathcal{V}$ which satisfies (A) and (or) (B) the quotient uniformity for Y relative to f, Y a quotient uniform space relative to f, and f a uniform quotient map. (See also [**1**, p. 150].)

The following theorem is concerned with the possibility that $f[\mathcal{U}]=\{f[U]\mid U\in\mathcal{U}\}$ be a uniformity.

THEOREM 1. *Let $\mathcal{U}$ be a uniformity for X, and let Δ be the diagonal of $Y\times Y$. Then*

(a) *If $f[\mathcal{U}]$ is a uniformity, then* (B) *is true when $\mathcal{V}=f[\mathcal{U}]$. In particular, $f[\mathcal{U}]$ is then the quotient uniformity for Y relative to f.*

(b) *$f[\mathcal{U}]$ lacks at most the "triangle axiom" to be a uniformity for Y. That is, $f[\mathcal{U}]$ is a uniformity if for each $V\in\mathcal{U}$ there exists $U\in\mathcal{U}$ such that $f[U]\circ f[U]\subset f[V]$.*

(c) *A necessary and sufficient condition for $f[\mathcal{U}]$ to be a uniformity is that for each $V\in\mathcal{U}$ there exists $U\in\mathcal{U}$ such that $U\circ f^{-1}[\Delta]\circ U\subset f^{-1}\circ f[V]$.*

(d) *If $\mathcal{U}$ is defined by a single pseudo-metric d, then a necessary and sufficient condition for $f[\mathcal{U}]$ to be a uniformity is that for each $\epsilon>0$ there exists $\delta>0$ such that*

$$d(x, R[z]) < \delta,\ d(y, R[z]) < \delta,\ x, y, z \in X \textit{ imply } d(R[x], R[y]) < \epsilon.$$

Here, and throughout this paper, the same symbol is used to denote the distance between sets or between a point and a set that is used to denote the distance between points.

PROOF. (a) and (b) are trivial. To prove (d), first observe that $f^{-1}\circ f[V]=f^{-1}[\Delta]\circ V\circ f^{-1}[\Delta]$ for any V and then verify in a straightforward manner that the condition in (d) is equivalent to the condition in (c) when $\mathcal{U}$ is defined by d. It remains to prove (c). So suppose that $f[\mathcal{U}]$ is a uniformity and let $V\in\mathcal{U}$. Then there exists $U\in\mathcal{U}$ such that $f[U]\circ f[U]\subset f[V]$. It follows that $U\circ f^{-1}[\Delta]\circ U\subset f^{-1}\circ f[V]$. For let $(u, v)\in U\circ f^{-1}[\Delta]\circ U$. Then there exists $(x, y)\in f^{-1}[\Delta]$ such that (u, x) and (y, v) are in U. Since $f(x)=f(y)$, we have $(f(u), f(x))\in f[U]$, $(f(x), f(v))\in f[U]$, and $f(u, v)\in f[U]$

$\circ \boldsymbol{f}[U] \subset \boldsymbol{f}[V]$. Thus the condition in (c) is necessary. To show that it is sufficient we need only verify the "triangle axiom" for $\boldsymbol{f}[\mathfrak{U}]$. Let $V \in \mathfrak{U}$ and let U be as in the condition of (c). Then $\boldsymbol{f}[U] \circ \boldsymbol{f}[U] \subset \boldsymbol{f}[V]$. For let $(u, v) \in \boldsymbol{f}[U] \circ \boldsymbol{f}[U]$, and let z be such that (u, z) and (z, v) are in $\boldsymbol{f}[U]$. Choose u_1, z_1, z_2, v_1 such that

$$(u_1, z_1) \in U, \qquad (f(u_1), f(z_1)) = (u, z),$$
$$(z_2, v_1) \in U, \qquad (f(z_2), f(v_1)) = (z, v).$$

It follows that

$$(u_1, v_1) \in U \circ \boldsymbol{f}^{-1}[\Delta] \circ U \subset \boldsymbol{f}^{-1} \circ \boldsymbol{f}[V],$$

and

$$(u, v) = \boldsymbol{f}(u_1, v_1) \in \boldsymbol{f}[V].$$

This completes the proof of Theorem 1.

If $\mathfrak{U}$ is a uniformity for X, it is not necessarily true that $\boldsymbol{f}[\mathfrak{U}]$ is a uniformity for Y. For example, let f be the map from the unit interval onto the unit circle defined by $f(x) = e^{2\pi i x}$, for $0 \leqq x \leqq 1$. f is a uniform quotient map, and in fact the uniformity of the circle satisfies condition (B). However, it is easy to prove that the condition in (d) of Theorem 1 is not satisfied for any metric compatible with the uniformity of the interval. In the general case, the quotient uniformity for Y is the image of a uniformity for X smaller than the original one. In fact, if $\mathfrak{U}$, $\mathfrak{V}$ are uniformities for X, Y, respectively, making f a uniform quotient map, then the uniformity $\mathfrak{W}$ for X with $\boldsymbol{f}^{-1}[\mathfrak{V}] = \{\boldsymbol{f}^{-1}[V] \mid V \in \mathfrak{V}\}$ as base is smaller than $\mathfrak{U}$ and $\boldsymbol{f}[\mathfrak{W}] = \mathfrak{V}$.

The next theorem relates quotient topologies and quotient uniformities (unfortunately, in a very special case).

THEOREM 2. *Let $\mathfrak{U}$ be a uniformity for X such that $\boldsymbol{f}[\mathfrak{U}]$ is a uniformity for Y, give X the topology of $\mathfrak{U}$, and suppose that $f^{-1}[y]$ is compact for each $y \in Y$. Then the quotient topology on Y relative to f is the topology of the quotient uniformity $\boldsymbol{f}[\mathfrak{U}]$.*

PROOF. Let G be an open set in the quotient topology on Y, and let $y \in G$. Since $f^{-1}[G]$ is a neighborhood of the compact set $f^{-1}[y]$, there exists $U \in \mathfrak{U}$ such that $U[f^{-1}[y]] \subset f^{-1}[G]$. It follows that $\boldsymbol{f}[U][y] \subset G$. For let $z \in \boldsymbol{f}[U][y]$. Then $(y, z) \in \boldsymbol{f}[U]$, say $(y, z) = (f(u), f(v))$, with $(u, v) \in U$. Thus $v \in U[u] \subset U[f^{-1}[y]] \subset f^{-1}[G]$, and $z = f(v) \in G$. We have shown that G is a neighborhood of y in the topology of $\boldsymbol{f}[\mathfrak{U}]$, and hence that the quotient topology is smaller than the topology of the quotient uniformity $\boldsymbol{f}[\mathfrak{U}]$. Since the quotient topology is the largest topology making f continuous, the theorem follows.

In the previous theorem we may replace the assumption that each $f^{-1}(y)$ be compact with the assumption that, for each member G of some base for the quotient topology on Y and for each $y \in G$, there exists $U \in \mathfrak{U}$ such that $U[f^{-1}[y]] \subset f^{-1}[G]$. However, the heavy restrictions on $\mathfrak{U}$ seem unavoidable. For let X be the subset of the plane consisting of points $a=(0, 0)$, $b=(0, 1)$, and segments C_n joining $(1/n, 0)$ to $(1/n, 1)$ for $n=1, 2, \cdots$; let Y consist of a, b, and the points $a_n=(1/n, 0)$ for $n=1, 2, \cdots$; and define $f: X \to Y$ by $f(a)=a$, $f(b)=b$, $f[C_n]=a_n$ for $n=1, 2, \cdots$. Give X the uniformity and topology induced by the usual plane metric; then $f^{-1}(y)$ is compact for all $y \in Y$. If $\mathfrak{V}$ is the quotient uniformity on Y (or any uniformity which makes f uniformly continuous), then $(a, b) \in \bigcap \mathfrak{V}$, and so in the induced topology a, b have the same neighborhoods. But, in the quotient topology on Y, $Y - \{b\}$ is a neighborhood of a and not of b.

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