PRESERVATION OF PSEUDO-METRIZABILITY BY QUOTIENT MAPS[1]

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In this paper we obtain necessary and sufficient conditions for a quotient map to preserve pseudo-metrizability. All notation and terminology not explicitly developed here are as in [**1**]. We assume throughout that f is a given function from a set X onto a set Y. Moreover, we will need the following theorem, whose proof is the same as the proof of Theorem 2 in [**1**].

THEOREM 1. *Let* $\mathfrak{U}$ *be a uniformity for* X *such that* $\boldsymbol{f}(\mathfrak{U})$ *is a uniformity for* Y, *and give* X *the topology of* $\mathfrak{U}$. *Then the quotient topology on* Y *relative to* f *is the topology of the quotient uniformity* $\boldsymbol{f}(\mathfrak{U})$ *if, for each member* G *of some base of the quotient topology on* Y *and for each* $y \in G$, *there exists* $U \in \mathfrak{U}$ *such that* $U[f^{-1}[y]] \subset f^{-1}[G]$.

Recall that a uniformity $\mathfrak{U}$ for X can be defined by a single pseudo-metric if and only if $\mathfrak{U}$ has a countable base. Moreover, $\boldsymbol{f}(\mathfrak{U})$ has a countable base whenever $\mathfrak{U}$ does. Hence we obtain immediately the following.

THEOREM 2. *Suppose that* X *is a pseudo-metrizable space and that there is a pseudo-metric and corresponding uniformity* $\mathfrak{U}$ *for* X *which yields the given topology on* X *and is such that* $\boldsymbol{f}(\mathfrak{U})$ *is a uniformity. Suppose further that, for each member* G *of some base of the quotient topology on* Y *and for each* $y \in G$, *there exists* $U \in \mathfrak{U}$ *such that* $U[f^{-1}[y]] \subset f^{-1}[G]$. *(This last condition is satisfied if, for example,* $f^{-1}[y]$ *is compact for each* $y \in G$.) *Then the quotient topology on* Y *relative to* f *is pseudo-metrizable.*

Our intention now is to obtain a theorem on preservation of pseudo-metrizability which does not require that $\boldsymbol{f}[\mathfrak{U}]$ be a uniformity. We will need some notational conventions. If (W, d) is a pseudo-metric space, d is allowed to take the value ∞, since any such pseudo-metric can be replaced by a bounded one without changing the underlying uniform structure. If g is a function from W to Y, $y \in Y$, and $\epsilon > 0$, then $N_\epsilon[g^{-1}[y]] = \{x \in W \mid d(x, g^{-1}[y]) < \epsilon\}$. If, in

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addition, Y has a topology, we say that W, d, Y, g satisfy condition (C) if and only if the following condition is satisfied.

(C) For each member G of some base of Y there exists a family $\{\epsilon(y) \mid y \in G\}$ of positive real numbers such that

$$\text{(i)} \qquad N_{\epsilon(y)}[g^{-1}[y]] \subset g^{-1}[G], \quad \text{if } y \in G,$$

and

$$\text{(ii)} \qquad d(g^{-1}[y], g^{-1}[z]) \geqq \epsilon(y) - \epsilon(z), \quad \text{if } y, z \in G.$$

THEOREM 3. *Let X be a pseudo-metrizable space. Then the quotient topology on Y relative to f is pseudo-metrizable if and only if the topology on X can be defined by a pseudo-metric d such that X, d, Y, f satisfy* (C).

PROOF. Suppose that the quotient topology on Y is pseudo-metrizable, say by a bounded pseudo-metric d_Y, and let d_X be any pseudo-metric consistent with the topology of X. Define $d(x, y) = d_X(x, y) + d_Y(f(x), f(y))$ for $x, y \in X$. Then d is a pseudo-metric on X, giving the same topology as d_X. It is now easy to see that X, d, Y, f satisfy (C). For if G is any proper open subset of Y, let $\epsilon(y) = d_Y(y, Y - G)$ for each $y \in G$, and then verify (i) and (ii) of (C).

On the other hand, suppose that Y has the quotient topology and that d is a pseudo-metric for X such that X, d, Y, f satisfy (C). In the sequence of propositions to follow this proof we will construct a pseudo-metrizable space X_∞ containing X, a pseudo-metric d_∞ defining the topology of X_∞, and an extension of f to a quotient map $f_\infty: X_\infty \to Y$ such that: for each G in the base described in (C) and each $y \in G$ there exists $\epsilon > 0$ for which $N_\epsilon[f_\infty^{-1}[y]] \subset f_\infty^{-1}[G]$; and $\mathbf{f}_\infty[\mathcal{U}_\infty]$ is a uniformity, where $\mathbf{f}_\infty$ is defined by $\mathbf{f}_\infty(x, y) = (f_\infty(x), f_\infty(y))$ for $x, y \in X_\infty$ and $\mathcal{U}_\infty$ is the uniformity defined by d_∞. It then follows from Theorem 2 that the quotient topology on Y is pseudo-metrizable.

It remains to construct X_∞, d_∞, f_∞ given X, d, f.

For each ordered pair (u, v) of points of X such that $f(u) = f(v)$, let ϕ be an isometry from X onto a space $\phi[X]$ whose intersection with X is v and which is such that $\phi(u) = v$. Let Φ be the set of isometries chosen in this way. For each $X \cup \phi[X]$ define a pseudo-metric d_ϕ by

$$\begin{aligned} d_\phi(x, y) &= d(x, y), && \text{if } x, y \in X, \\ &= d(\phi^{-1}[x], \phi^{-1}[y]), && \text{if } x, y \in \phi[X], \\ &= d(x, v) + d(u, \phi^{-1}[y]), && \text{if } x \in X, y \in \phi[X], \\ &= d(\phi^{-1}[x], u) + d(v, y), && \text{if } x \in \phi[X], y \in X, \end{aligned}$$

and extend f to the function $f_\phi: X \cup \phi[X] \to Y$ defined by

$$f_\phi(x) = f(x), \qquad \text{if } x \in X,$$
$$= f \circ \phi^{-1}[x], \qquad \text{if } x \in \phi[X].$$

Next regard any two of the spaces $X \cup \phi[X]$, $X \cup \psi[X]$, with ϕ, ψ distinct members of Φ, as disjoint, and let

$$X_1 = \bigcup \{ X \cup \phi[X] \mid \phi \in \Phi \}.$$

Define a pseudo-metric d_1 for X_1 by

$$d_1(x, y) = d_\phi(x, y), \qquad \text{if } x, y \in X \cup \phi[X], \phi \in \Phi,$$
$$= \infty, \qquad \text{otherwise,}$$

and extend all f_ϕ's simultaneously to the function

$$f_1 = \bigcup \{ f_\phi \mid \phi \in \Phi \} : X_1 \to Y.$$

We regard X_1 as disjoint from X, although we keep in mind that X_1 contains Φ copies of X, on each of which f_1, d_1 behave exactly like f, d, respectively.

Obtain a sequence of mutually disjoint pseudo-metric spaces $(X, d) = (X_0, d_0)$, (X_1, d_1), $\cdots$, (X_n, d_n), $\cdots$ and functions $f_0 = f$: $X \to Y$, f_1: $X_1 \to Y$, $\cdots$, f_n: $X_n \to Y$, $\cdots$, by constructing Φ_n, X_{n+1}, d_{n+1}, f_{n+1} from X_n, d_n, f_n in the same way that Φ, X_1, d_1, f_1 were constructed from X, d, f. If $m < n$, then X_n contains $\Phi_m \times \cdots \times \Phi_{n-1}$ copies of X_m, on each of which f_n, d_n behave exactly like f_m, d_m, respectively. Now define

$$X_\infty = \bigcup \{ X_n \mid n = 0, 1, \cdots \}, \qquad f_\infty = \bigcup \{ f_n \mid n = 0, 1, \cdots \},$$
$$d_\infty(x, y) = d_n(x, y), \qquad \text{if } x, y \in X_n,$$
$$= \infty, \qquad \text{if } x \in X_m, y \in X_n, m \neq n.$$

In the propositions to follow we need the notation defined below.

$$R_n[x] = \{ z \in X_n \mid f_n(z) = f_\infty(x) \}, \qquad \text{if } x \in X_\infty,$$
$$R_\infty[x] = \{ z \in X_\infty \mid f_\infty(z) = f_\infty(x) \}, \qquad \text{if } x \in X_\infty.$$

Observe that

$$R_0[x] = R[x], \qquad \text{if } x \in X,$$
$$R_\infty[x] = \bigcup \{ R_n[x] \mid n = 0, 1, \cdots \}, \qquad \text{if } x \in X_\infty.$$

PROPOSITION 1. *Suppose* $\epsilon > 0$ *and that* x, y, $z \in X_\infty$ *are such that* $d_\infty(x, R_n[z]) < \epsilon/2$ *and* $d_\infty(y, R_n[z]) < \epsilon/2$. *Then* $d_\infty(R_{n+1}[x], R_{n+1}[y]) < \epsilon$.

PROOF. Clearly, x, $y \in X_n$. Let u, $v \in R_n[z]$ be such that $d_n(x, u)$

$<\epsilon/2$, $d_n(y, v)<\epsilon/2$, and let $\phi\in\Phi_n$ be such that $\phi(u)=v$. Then $\phi(x)\in R_{n+1}[x]$, and we let y represent both itself as a point of X_n and the point of $X_n\cup\phi[X_n]$ which corresponds to it under the identity injection. It follows that

$$d_\infty(R_{n+1}[x], R_{n+1}[y]) = d_{n+1}(R_{n+1}[\phi(x)], R_{n+1}[y]) \leqq d_{n+1}(\phi(x), y)$$
$$= d_n(y, v) + d_n(x, u) < \epsilon.$$

PROPOSITION 2. *Suppose* $\epsilon>0$ *and that* $x, y, z\in X_\infty$ *are such that* $d_\infty(x, R_\infty[z])<\epsilon/2$, $d_\infty(y, R_\infty[z])<\epsilon/2$. *Then* $d_\infty(R_\infty[x], R_\infty[y])<\epsilon$.

PROOF. There exist m, n such that $x\in X_m$, $y\in X_n$. Hence $d_\infty(x, R_m[z])<\epsilon/2$ and $d_\infty(y, R_n[z])<\epsilon/2$. Suppose $m\leqq n$. By an easy inductive argument it follows that there exists $w\in R_n[x]\subset R_\infty[x]$ such that $d_\infty(w, R_n[z])<\epsilon/2$. So, by Proposition 1, we have

$$d_\infty(R_\infty[x], R_\infty[y]) \leqq d_\infty(R_{n+1}[w], R_{n+1}[y]) < \epsilon.$$

PROPOSITION 3. *If* $\mathcal{U}_\infty$ *is the uniformity for* X_∞ *defined by* d_∞, *then* $f_\infty[\mathcal{U}_\infty]$ *is a uniformity.*

PROOF. Apply Proposition 2 to (d) of Theorem 1 in [1].

PROPOSITION 4. *Suppose* X, d, Y, f *satisfy condition* (C). *If* $n=0, 1, \cdots$, *then* X_n, d_n, Y, f_n *also satisfy* (C), *with, for each basic open subset* G *of* Y, *the same choice of* $\{\epsilon(y)\mid y\in G\}$ *that is used for* X, d, Y, f.

PROOF. By induction. The proposition is trivially true for $n=0$. The argument establishing the induction step is the same, except for notation, as the proof that X_1, d_1, Y, f_1 satisfy (C). This being the case, we give here only the latter.

Observe that, since

$$N_\epsilon[\overset{-1}{f_1}[y]] = \bigcup\{N_\epsilon[\overset{-1}{f_\phi}[y]] \mid \phi\in\Phi\}, \qquad \text{if } \epsilon>0 \text{ and } y\in Y,$$

and

$$d_1(\overset{-1}{f_1}[y], \overset{-1}{f_1}[z]) = \inf\{d_\phi(\overset{-1}{f_\phi}[y], \overset{-1}{f_\phi}[z]) \mid \phi\in\Phi\}, \qquad \text{if } y, z\in Y,$$

it is clearly sufficient to show that $X\cup\phi[X]$, d_ϕ, Y, f_ϕ satisfy (C) with, for each basic open subset G of Y, the same choice of $\{\epsilon(y)\mid y\in G\}$ that works for X, d, Y, f.

So let $u, v\in X$ be such that $f(u)=f(v)=w$, and let ϕ be the isometry of X such that $\phi(u)=v$. Let G be a basic open subset of Y, and let $\{\epsilon(y)\mid y\in G\}$ be chosen as in (C) applied to X, d, Y, f. Then it remains to prove

(i) $$N_{\epsilon(y)}[\overset{-1}{f_\phi}[y]] \subset \overset{-1}{f_\phi}[G], \qquad \text{if } y \in G,$$

and

(ii) $$d_\phi(\overset{-1}{f_\phi}[y], \overset{-1}{f_\phi}[z]) \geqq \epsilon(y) - \epsilon(z), \qquad \text{if } y, z \in G.$$

To prove (i), suppose $d_\phi(x, f_\phi^{-1}[y]) < \epsilon(y)$, $y \in G$. Since $f_\phi^{-1}[y] = f^{-1}[y] \cup \phi \circ f^{-1}[y]$, we consider four cases:

Case 1. $x \in X$, $d(x, f^{-1}[y]) < \epsilon(y)$,

Case 2. $x \in X$, $d_\phi(x, \phi \circ f^{-1}[y]) < \epsilon(y)$,

Case 3. $x \in \phi[X]$, $d_\phi(x, \phi \circ f^{-1}[y]) < \epsilon(y)$,

Case 4. $x \in \phi[X]$, $d_\phi(x, f^{-1}[y]) < \epsilon(y)$.

Since $f_\phi^{-1}[G] = f^{-1}[G] \cup \phi \circ f^{-1}[G]$, it follows easily from Case 1 or Case 3 that $x \in f_\phi^{-1}[G]$. Cases 2 and 4 are similar, and we give the details only for Case 2. We have in this case

$$d_\phi(x, \phi \circ f^{-1}[y]) = d(x, v) + d(u, f^{-1}[y]) < \epsilon(y).$$

Hence $u \in f^{-1}[G]$ and $w = f(u) \in G$. Moreover,

$$d(x, v) < \epsilon(y) - d(u, f^{-1}[y]) \leqq \epsilon(y) - d(f^{-1}[w], f^{-1}[y]) \leqq \epsilon(w),$$

so that $x \in N_{\epsilon(w)}[f^{-1}[w]] \subset f^{-1}[G] \subset f_\phi^{-1}[G]$.

To prove (ii), let $y, z \in G$, and observe that $d_\phi(f_\phi^{-1}[y], f_\phi^{-1}[z])$ is the minimum of the four quantities $d(f^{-1}[y], f^{-1}[z])$, $d_\phi(\phi \circ f^{-1}[y], \phi \circ f^{-1}[z])$, $d_\phi(f^{-1}[y], \phi \circ f^{-1}[z])$, $d_\phi(\phi \circ f^{-1}[y], f^{-1}[z])$. We consider only the case in which

$$d_\phi(f_\phi^{-1}[y], f_\phi^{-1}[z]) = d_\phi(f^{-1}[y], \phi \circ f^{-1}[z]),$$

the others being either similar or trivial. If $w \in G$, we have

$$\begin{aligned} d_\phi(f^{-1}[y], \phi \circ f^{-1}[z]) &= d(f^{-1}[y], v) + d(u, f^{-1}[z]) \\ &\geqq d(f^{-1}[y], f^{-1}[w]) + d(f^{-1}[w], f^{-1}[z]) \\ &\geqq \epsilon(y) - \epsilon(z). \end{aligned}$$

If $w \notin G$, there is no point of $f^{-1}[w]$ in either $V_{\epsilon(y)}[f^{-1}[y]]$ or in $V_{\epsilon(z)}[f^{-1}[z]]$. Hence,

$$\begin{aligned} d_\phi(f^{-1}[y], \phi \circ f^{-1}[z]) &\geqq d(f^{-1}[y], f^{-1}[w]) + d(f^{-1}[w], f^{-1}[z]) \\ &\geqq \epsilon(y) + \epsilon(z) \geqq \epsilon(y) - \epsilon(z). \end{aligned}$$

PROPOSITION 5. *If* X, d, Y, f *satisfy* (C), *then so also do* X_∞, d_∞, Y_∞, f_∞, *with, for each basic open subset* G *of* Y, *the same choice of* $\{\epsilon(y) \mid y \in G\}$ *that is used for* X, d, Y, f.

PROOF. This proposition follows from the previous one. For let G

be a basic open subset of Y and let $\{\epsilon(y) \mid y \in G\}$ be as in (C) applied to X, d, Y, f. Then

$$N_{\epsilon(y)}[f_\infty^{-1}[y]] = \bigcup\{N_{\epsilon(y)}[f_n^{-1}[y]] \mid n = 0, 1, \cdots\} \subset f_\infty^{-1}[G], \quad \text{if } y \in G,$$

and

$$d_\infty(f_\infty^{-1}[y], f_\infty^{-1}[z]) = \inf\{d_n(f_n^{-1}[y], f_n^{-1}[z]) \mid n = 0, 1, \cdots\}$$
$$\geqq \epsilon(y) - \epsilon(z), \quad \text{if } y, z \in G.$$

PROPOSITION 6. *If Y has a topology which makes f a map (a quotient map), then f_∞ is a map (a quotient map).*

PROOF. Trivial.

This completes the material needed to prove Theorem 3. In fact, we need from Proposition 5 only the fact that X_∞, d_∞, Y, f_∞ satisfy part (i) of (C). Part (ii) is needed for X, d, Y, f to maintain part (i) in the proof of Proposition 4. It is also worth noting that (C) is not needed in Propositions 3 and 6.

In [2] and [3] it is shown that, if X is metrizable and f is a closed map, then Y is metrizable if $\mathrm{B}df^{-1}[y]$ is compact for all $y \in Y$. This same result is true for pseudo-metrizable spaces if Y is T_1 or regular. Hence, if X is pseudo-metrizable, Y is T_1 or regular, f is a closed map, and each $\mathrm{B}df^{-1}[y]$ is compact, it follows from the "only if" part of Theorem 3 that X, d, Y, f satisfy (C) for some pseudo-metric d compatible with X. It would be interesting to find directly a pseudo-metric d such that X, d, Y, f satisfy (C), and hence obtain a new proof of the result in [2] and [3]. It would be even more desirable to show (if such be the case) that X, d, Y, f satisfy (C) for every pseudo-metric d compatible with X. The best this author has been able to manage in this direction is the following.

THEOREM 4. *Suppose that X is pseudo-metrizable with pseudo-metric d, that Y is a topological space, and that f is continuous (f need not be closed). Then X, d, Y, f satisfy* (C) *if there is a finite subset A of Y such that $f^{-1}[y]$ is compact for all $y \in A$ and* $\operatorname{diam} f^{-1}[y] = 0$ *for all $y \in Y - A$.*

PROOF. Let A be as in the theorem, and let G be an arbitrary open subset of Y. For each $y \in G$, let $\delta(y) > 0$ be the largest real number such that $N_{\delta(y)}[f^{-1}[y]] \subset f^{-1}[G]$. Let $B = A \cap G$, and let $\epsilon > 0$ be such that $\epsilon < \delta(y)/2$ for all $y \in B$. Finally, define

$$\epsilon(y) = \epsilon, \qquad \text{if } f^{-1}[y] \subset N_\epsilon[f^{-1}[B]],$$
$$= \min\{\epsilon, \delta(y)\}, \qquad \text{if } f^{-1}[y] \not\subset N_\epsilon[f^{-1}[B]].$$

It is immediate that $\{\epsilon(y) \mid y \in G\}$ satisfies part (i) of (C). To check part (ii), let y, $z \in G$ and consider the following cases.

Case 1. $f^{-1}[y]$, $f^{-1}[z] \subset N_\epsilon[f^{-1}[B]]$. Then

$$\epsilon(y) - \epsilon(z) = 0 \leqq d(f^{-1}[y], f^{-1}[z]).$$

Case 2. $f^{-1}[y] \not\subset N_\epsilon[f^{-1}[B]]$ and $f^{-1}[z] \not\subset N_\epsilon[f^{-1}[B]]$. Then diam $f^{-1}[y] = 0 =$ diam $f^{-1}[z]$, and it follows easily that $\epsilon(y) - \epsilon(z) \leqq d(f^{-1}[y], f^{-1}[z])$.

Case 3. $f^{-1}[y] \not\subset N_\epsilon[f^{-1}[B]]$, $f^{-1}[z] \subset N_\epsilon[f^{-1}[B]]$. Hence diam $f^{-1}[y] = 0$. Observe that $\epsilon - d(f^{-1}[y], f^{-1}[z]) \leqq \delta(y)$, since

$$d(u, f^{-1}[y]) < \epsilon - d(f^{-1}[y], f^{-1}[z]) \Rightarrow d(u, f^{-1}[z]) < \epsilon \Rightarrow d(u, f^{-1}[B]) < 2\epsilon \text{ (since } f^{-1}[z] \subset N_\epsilon[f^{-1}[B]]) \Rightarrow u \in f^{-1}[G].$$

Now, using the inequality $\epsilon - \delta(y) \leqq d(f^{-1}[y], f^{-1}[z])$, it is easy to check that $\epsilon(z) - \epsilon(y) \leqq d(f^{-1}[z], f^{-1}[y])$. The inequality $\epsilon(y) - \epsilon(z) \leqq d(f^{-1}[y], f^{-1}[z])$ is trivial.

If A is not finite in the preceding theorem, one apparently must use the hypothesis that f is closed to prove (C), since, in the example following Theorem 2 of [**1**], f is open and continuous, X is metrizable, and each $f^{-1}[y]$ is compact, but Y is not normal and hence not pseudo-metrizable. In [**3**, p. 699] there is an example in which f is open and continuous, X is metrizable, each $f^{-1}[y]$ is compact, and Y is Hausdorff but not normal.

On the other hand, the assumption that f is a quotient map and that (C) is true implies neither that f is closed nor that each $f^{-1}[y]$ is compact. Hence Theorem 3 is not contained in [**2**] and [**3**]. For an example, consider the usual projection of the plane onto the reals.

With regard to the question preceding Theorem 4, it would be interesting to know, whether the truth of (C) for some d implies (C) for all equivalent (or even uniformly equivalent) pseudo-metrics, under the more general hypothesis that f is a quotient map rather than a closed map.

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