ARTINIAN AND NOETHERIAN HYPERCENTRAL GROUPS

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1. Recently, Kemhadze [1] has proved that a finite group G is nilpotent if and only if each nonabelian subgroup S of G has a noncyclic commutator factorgroup. In this note we will generalize Kemhadze's theorem in two ways: the first two theorems will be concerned with artinian groups which contain Kemhadze's theorem as a special case; our third theorem will characterize noetherian nilpotent groups in a similar way.

I want to take this opportunity to express my appreciation to Professoı Reinhold Baer for his interest in the completion of this paper.

2. **Notations.**

artinian = minimum condition for subgroups.

noetherian = maximum condition for subgroups.

factor of G = any epimorphic image of any subgroup of G.

$Z(G)$ = center of G.

$Z_0=1\leqq Z_1=Z(G)\leqq \cdots \leqq Z_\alpha \leqq \cdots$ are the terms of the upper central series of G (possibly continued transfinitely).

hypercenter = last term of the upper central series.

hypercentral i.e. G itself is a term of its upper central series.

nilpotent i.e. $G=Z_n$, n a natural number.

class of G = smallest integer n (provided that it does exist) such that $G=Z_n$.

$x \circ y = x^{-1}y^{-1}xy$.

$A \circ B$ = subgroup of G generated by $a \circ b$ where $a \in A$ and $b \in B$.

${}^0G=G\geqq {}^1G=G\circ G\geqq {}^2G=G\circ {}^1G\geqq \cdots \geqq {}^\alpha G\geqq \cdots$ are the terms of the lower central series (possibly continued transfinitely).

p = natural prime number.

3. In the following lemma $\mathfrak{E}$ will be a group-theoretical property such that if A and B are two normal $\mathfrak{E}$-subgroups of the group G then AB is also a (normal) $\mathfrak{E}$-subgroup of G.

LEMMA. *If M is a group which is not an $\mathfrak{E}$-group but all whose proper normal subgroups are $\mathfrak{E}$-groups, then M/M' is either a finite cyclic p-group or a Prüfer group of type p^∞.*

PROOF. Since the product of any two proper normal subgroups of

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M is a proper subgroup of M, the result follows by Newman-Wiegold [**1**, p. 244].

In Theorems 1 and 2 below we will apply this lemma in case $\mathfrak{E}=$ hypercentral (according to P. Hall [**1**, Lemma 1, p. 334] "hypercentral" meets the requirement for $\mathfrak{E}$) and $\mathfrak{E}=$nilpotent.

THEOREM 1. *The following properties of the artinian group G are equivalent*:

(I) *G is hypercentral.*

(II) *If S is a nonabelian finitely generated subgroup of G and if S/S' is primary, then S/S' is noncyclic.*

PROOF. (I)$\Rightarrow$(II). According to Baer [**3**, Satz **4.1**, p. 21] G is locally finite-and-nilpotent; hence (II) is a consequence of (I).

(II)$\Rightarrow$(I). Deny. Since G is artinian, there exists a subgroup M of G which is not hypercentral but whose proper subgroups are hypercentral; in particular M is not abelian. If M were not finitely generated, each finitely generated subgroup F of M would be hypercentral. Since G and a fortiori F are artinian, F would be finite and nilpotent, so that the artinian group M is locally finite-and-nilpotent and therefore by Baer [**3**, Satz **4.1**, p. 21] M is hypercentral, a contradiction. Therefore M is finitely generated. According to the preceding lemma, M/M' is a cyclic p-group. Since M is not abelian, this is a contradiction. Q.E.D.

After deleting "finitely generated" from (II) in Theorem 1 one obtains

THEOREM 2. *The following properties of the artinian group G are equivalent*:

(I) *G is nilpotent*

(II) *If S is a nonabelian subgroup of G and if S/S' is primary, then S/S' is noncyclic.*

PROOF. (I)$\Rightarrow$(II). Clear.

(II)$\Rightarrow$(I). Deny. Since G is artinian, there exists a subgroup M of G which is not nilpotent, but whose proper subgroups are nilpotent. By (II) M/M' is not cyclic so that according to the preceding lemma M/M' is a Prüfer group of type p^∞. By Theorem 1 the artinian group M is hypercentral; by Baer [**3**, Satz **4.1**, p. 21] there exists an abelian normal subgroup $A \lhd M$ with finite M/A. Since M/AM' is a finite factorgroup of the Prüfer group M/M', $M=AM'$. Since $M' \lhd M$, M' is nilpotent and hence $AM'=M$ is nilpotent, a contradiction. Q.E.D.

THEOREM 3. *G is noetherian and nilpotent if and only if* (a) *G is finitely generated,* (b) *each nonabelian factor of G has a noncyclic commutator factorgroup and* (c) *there exists an integer* $n \geqq 0$ *such that the class of a finite nilpotent factorgroup of G does not exceed n.*

PROOF. The necessity of (a), (b) and (c) is readily seen.

Now assume the validity of (a), (b) and (c) and deny that G is noetherian and nilpotent. Then by Baer [1, Lemma 4, p. 410] there exists an epimorphic image H of G which is not of finite class, but whose proper epimorphic images are of finite class.

(1) *H does not contain abelian normal subgroups* $\neq 1$.

Assume there exists an abelian normal subgroup $A \neq 1$ of H. Since H/A is finitely generated and nilpotent, by Baer [2, Satz 1, p. 310] H is nilpotent, a contradiction.

(2) ${}^{\omega}H = \bigcap_{i=0}^{\infty} {}^{i}H = 1$.

Deny. Apply (b) to see that $({}^{\omega}H)' \neq {}^{\omega}H \neq 1$ and apply (1) to show $({}^{\omega}H)' \neq 1$. Hence $H/({}^{\omega}H)'$ is of finite class, proving the existence of a positive integer c with the property ${}^{\omega}H \leqq {}^{c}H \leqq ({}^{\omega}H)' < {}^{\omega}H$, the desired contradiction.

(3) Since ${}^{\omega}H = \bigcap_{i=0}^{\infty} {}^{i}H = 1$, by Baer [2, p. 306] the intersection of all normal subgroups X of H with finite nilpotent factorgroup H/X is 1. By (c) ${}^{n}H \leqq X$ for all these X; therefore ${}^{n}H = 1$, i.e. H is nilpotent. By (a) and Baer [2, Satz B, p. 299] H is noetherian, i.e. H is noetherian and nilpotent, a contradiction. Q.E.D.

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