ARTINIAN AND NOETHERIAN HYPERCENTRAL GROUPS

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1. Recently, Kemhadze [1] has proved that a finite group $G$ is nilpotent if and only if each nonabelian subgroup $S$ of $G$ has a non-cyclic commutator factor group. In this note we will generalize Kemhadze's theorem in two ways: the first two theorems will be concerned with artinian groups which contain Kemhadze's theorem as a special case; our third theorem will characterize noetherian nilpotent groups in a similar way.

I want to take this opportunity to express my appreciation to Professor Reinhold Baer for his interest in the completion of this paper.

2. Notations.
    artinian = minimum condition for subgroups.
    noetherian = maximum condition for subgroups.
    factor of $G$ = any epimorphic image of any subgroup of $G$.
    $Z(G)$ = center of $G$.
    $Z_0 = 1 \leq Z_1 = Z(G) \leq \cdots \leq Z_\alpha \leq \cdots$ are the terms of the upper central series of $G$ (possibly continued transfinitely).
    hypercenter = last term of the upper central series.
    hypercentral i.e. $G$ itself is a term of its upper central series.
    nilpotent i.e. $G = Z_n$, $n$ a natural number.
    class of $G =$ smallest integer $n$ (provided that it does exist) such that $G = Z_n$.
    $x \circ y = x^{-1}y^{-1}xy$.
    $A \circ B =$ subgroup of $G$ generated by $a \circ b$ where $a \in A$ and $b \in B$.
    $^0G = G \geq ^1G = G \circ G \geq ^2G = G \circ ^1G \geq \cdots \geq ^nG \geq \cdots$ are the terms of the lower central series (possibly continued transfinitely).
    $p =$ natural prime number.

3. In the following lemma $\mathcal{E}$ will be a group-theoretical property such that if $A$ and $B$ are two normal $\mathcal{E}$-subgroups of the group $G$ then $AB$ is also a (normal) $\mathcal{E}$-subgroup of $G$.

    **Lemma.** If $M$ is a group which is not an $\mathcal{E}$-group but all whose proper normal subgroups are $\mathcal{E}$-groups, then $M/M'$ is either a finite cyclic $p$-group or a Prüfer group of type $p^\infty$.

    **Proof.** Since the product of any two proper normal subgroups of $M$ is a subgroup of $M$.

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$M$ is a proper subgroup of $M$, the result follows by Newman-Wiegold [1, p. 244].

In Theorems 1 and 2 below we will apply this lemma in case $G$ is hypercentral (according to P. Hall [1, Lemma 1, p. 334] “hypercentral” meets the requirement for $G$) and $G$ is nilpotent.

**Theorem 1.** The following properties of the artinian group $G$ are equivalent:

(I) $G$ is hypercentral.

(II) If $S$ is a nonabelian finitely generated subgroup of $G$ and if $S/S'$ is primary, then $S/S'$ is noncyclic.

**Proof.** (I)⇒(II). According to Baer [3, Satz 4.1, p. 21] $G$ is locally finite-nilpotent; hence (II) is a consequence of (I).

(II)⇒(I). Deny. Since $G$ is artinian, there exists a subgroup $M$ of $G$ which is not hypercentral but whose proper subgroups are hypercentral; in particular $M$ is not abelian. If $M$ were not finitely generated, each finitely generated subgroup $F$ of $M$ would be hypercentral. Since $G$ and a fortiori $F$ are artinian, $F$ would be finite and nilpotent, so that the artinian group $M$ is locally finite-nilpotent and therefore by Baer [3, Satz 4.1, p. 21] $M$ is hypercentral, a contradiction. Therefore $M$ is finitely generated. According to the preceding lemma, $M/M'$ is a cyclic $p$-group. Since $M$ is not abelian, this is a contradiction. Q.E.D.

After deleting “finitely generated” from (II) in Theorem 1 one obtains

**Theorem 2.** The following properties of the artinian group $G$ are equivalent:

(I) $G$ is nilpotent

(II) If $S$ is a nonabelian subgroup of $G$ and if $S/S'$ is primary, then $S/S'$ is noncyclic.

**Proof.** (I)⇒(II). Clear.

(II)⇒(I). Deny. Since $G$ is artinian, there exists a subgroup $M$ of $G$ which is not nilpotent, but whose proper subgroups are nilpotent. By (II) $M/M'$ is not cyclic so that according to the preceding lemma $M/M'$ is a Prüfer group of type $p^\infty$. By Theorem 1 the artinian group $M$ is hypercentral; by Baer [3, Satz 4.1, p. 21] there exists an abelian normal subgroup $A\triangleleft M$ with finite $M/A$. Since $M/AM'$ is a finite factorgroup of the Prüfer group $M/M'$, $M = AM'$. Since $M' < M$, $M'$ is nilpotent and hence $AM' = M$ is nilpotent, a contradiction. Q.E.D.
Theorem 3. $G$ is noetherian and nilpotent if and only if (a) $G$ is finitely generated, (b) each nonabelian factor of $G$ has a noncyclic commutator factorgroup and (c) there exists an integer $n \geq 0$ such that the class of a finite nilpotent factorgroup of $G$ does not exceed $n$.

Proof. The necessity of (a), (b) and (c) is readily seen.

Now assume the validity of (a), (b) and (c) and deny that $G$ is noetherian and nilpotent. Then by Baer [1, Lemma 4, p. 410] there exists an epimorphic image $H$ of $G$ which is not of finite class, but whose proper epimorphic images are of finite class.

(1) $H$ does not contain abelian normal subgroups $\neq 1$.

Assume there exists an abelian normal subgroup $A \neq 1$ of $H$. Since $H/A$ is finitely generated and nilpotent, by Baer [2, Satz 1, p. 310] $H$ is nilpotent, a contradiction.

(2) $^{\ast}H = \bigcap_{i=0}^{\infty} iH = 1$.

Deny. Apply (b) to see that $(^{\ast}H)' \neq ^{\ast}H \neq 1$ and apply (1) to show $(^{\ast}H)' \neq 1$. Hence $H/(^{\ast}H)'$ is of finite class, proving the existence of a positive integer $c$ with the property $^{\ast}H \leq ^{c}H \leq (^{\ast}H)' < ^{\ast}H$, the desired contradiction.

(3) Since $^{\ast}H = \bigcap_{i=0}^{\infty} iH = 1$, by Baer [2, p. 306] the intersection of all normal subgroups $X$ of $H$ with finite nilpotent factorgroup $H/X$ is 1. By (c) $^{\ast}H \leq X$ for all these $X$; therefore $^{\ast}H = 1$, i.e. $H$ is nilpotent. By (a) and Baer [2, Satz B, p. 299] $H$ is noetherian, i.e. $H$ is noetherian and nilpotent, a contradiction. Q.E.D.

Bibliography

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