HOMOLOGY AND PRESENTATIONS OF ALGEBRAS

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Introduction. Epstein [2] has published certain results on deficiencies of finite presentations of groups. It is desired here to record analogues of some of those results for finite presentations of linear algebras. For this purpose it is relevant to define a certain homology theory for algebras (§1); thereafter the discussion is closely analogous to that in [2]. A connection between cohomology and a particular homology group related to deficiencies is considered in §3. Finally it is mentioned that these facts have analogues within "commutator" varieties of groups.

1. Homology of algebras. Let Λ denote some fixed commutative principal ideal domain. Let $V = V_S$ denote the variety of (not necessarily associative) Λ-algebras determined by a set S of identities, homogeneous in each variable and of total degree >1; e.g., the variety of associative, or Lie, or Jordan, algebras over Λ.

Every V-algebra A has a universal enveloping algebra $G_S(A)$, with the property that the right $G_S(A)$-modules are precisely the V-bimodules for A [4].

Following the (heuristic) "dual" of a procedure of Gerstenhaber [3] for defining *V-cohomology*, we may define *V-homology* modules $H_n(A; N)$ $[n \geqq 2]$ with coefficients in the left $G_S(A)$-module N as follows.

Let $A = F/R$ be a presentation of A as a quotient of a V-free algebra F, and consider the corresponding generic singular extension

$$0 \to R_* \xrightarrow{i} F_* \xrightarrow{\pi} A \to 0,$$

where $R_* = R/R^2$, $F_* = F/R^2$. Define

$$H_2(A; N) = \operatorname{Ker}\{i_*: N \otimes_{G'_S(A)} R_* \to N \otimes_\Lambda F_*/B_1(F_*, N)\},$$

where $G'_S(A)$ is the algebra opposite to $G_S(A)$ and $B_1(F_*, N)$ is the submodule of $N \otimes_\Lambda F_*$ generated by all elements

$$\pi y \cdot m \otimes z - m \otimes yz + m \cdot \pi z \otimes y \quad [m \in N; y, z \in F_*].$$

Also, define

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$$H_{n+2}(A;N) = \mathrm{Tor}_n^{G'_S(A)}(N, R_o) \quad [n > 0].$$

To see that this definition is independent of the presentation of A, note as in [3] that any $G_S(A)$-projective resolution of $R_\bullet$ may be combined with the extension $R_\bullet \to F_\bullet \to A$ to give a "generalized projective resolution" of A. The independence follows by standard techniques, and one obtains homology functors (normalized at degree 2) satisfying the usual axioms on existence of connected long exact sequences and the vanishing for degree >2 when N is $G'_S(A)$-projective.

If $G'_S(A)$ acts trivially on Λ, we note that

$$\begin{aligned} H_2(A;\Lambda) &\cong \mathrm{Ker}\{i_*: R_o/F_oR_o \cup R_oF_o \to F_o/F_o^2\} \\ &\cong F^2 \cap R/F \circ R, \end{aligned}$$

where $F \circ R$ is the ideal of F generated by $FR \cup RF$, i.e., the "Hopf isomorphism" is valid.

If Λ is a field and V is the variety of Lie (or of associative) Λ-algebras, the above definition gives the standard homology groups [1].

2. **Finite presentations.** Suppose that A has a *finite* presentation $P = \{x_1, \cdots, x_n/r_1, \cdots, r_m\}$ as a quotient of a V-free algebra F on generators $x_1, \cdots, x_n$ by the ideal R generated by relators $r_1, \cdots, r_m$. Define the *deficiency* of P to be

$$\mathrm{def}\, P = n - m.$$

PROPOSITION 2.1.

$$\mathrm{def}\, P \leqq \mathrm{rank}_\Lambda\, A/A^2 - gH_2(A;\Lambda),$$

where gT denotes the least possible number of generators for the finitely-generated module T.

PROOF. Consider the module extensions

$$0 \to F^2 \cap R/F \circ R \to R/F \circ R \to R/F^2 \cap R \to 0$$

and

$$0 \to (F^2 + R)/F^2 \to F/F^2 \to F/(F^2 + R) \to 0.$$

These give extensions

$$0 \to H_2(A;\Lambda) \to R/F \circ R \to T \to 0$$

and

$$0 \to T \to F/F^2 \to A/A^2 \to 0,$$

where $T = R/F^2 \cap R \cong (F^2 + R)/F^2$.

Since F/F^2 is Λ-free, the second sequence shows that T is Λ-free. Hence the first sequence splits, and

$$R/F \circ R = T + H_2(A;\Lambda).$$

Therefore

$$\begin{aligned} gH_2(A;\Lambda) &= g(R/F \circ R) - gT = g(R/F \circ R) - \operatorname{rank}_\Lambda T \\ &\leqq m - [n - \operatorname{rank}_\Lambda A/A^2]. \end{aligned}$$

Now call a V-algebra A *efficient* if equality in Proposition 2.1 is attainable.

EXAMPLE 2.2. If A is a finitely-generated zero algebra, then it is efficient.

This fact follows from the existence of canonical forms for finitely-generated Λ-modules.

EXAMPLE 2.3. If A has a finite V-presentation with precisely one relator, then A is efficient.

Here, $g(R/F \circ R) \neq 1$ would imply $R = F \circ R$. By induction, this would give $R \subseteq F^i$ (all i). Since every monomial of an element of F^i has total degree $\geqq i$ in $x_1, \cdots, x_n$, this is impossible for $R \neq \{0\}$.

Finally, by examining the standard presentation, one has

EXAMPLE 2.4. Let V be the variety of associative Λ-algebras, and let $P_n = \Lambda\{x_1, \cdots, x_n\}$ be a commutative polynomial ring without identity in n indeterminates over Λ. Then P_n is efficient, and

$$gH_2(P_n;\Lambda) = n(n-1)/2.$$

3. **Cohomology and H_2.** If M is a V-bimodule for a V-algebra A, let $H^n(A; M)$ denote the nth cohomology module of Gerstenhaber [**3**]. Let $G_S(A)$ act trivially on the Λ-module Γ defined as follows:

$$\begin{aligned} \Gamma &= \Lambda \qquad \text{if } \Lambda \text{ is a field,} \\ &= K/\Lambda \quad \text{otherwise,} \end{aligned}$$

where K is some fixed extension field of the field of fractions of Λ. Well known techniques of Eilenberg and MacLane give:

PROPOSITION 3.1. *There is a natural monomorphism*

$$\tau\colon H_2(A;\Lambda) \to \operatorname{Hom}_\Lambda(H^2(A;\Gamma), \Gamma).$$

PROOF. Given any presentation $A = F/R$ of A as in §1, there is an isomorphism

$$H^2(A; M) \cong \text{Hom}_A (R_o, M)/\text{Der} (F_o, M) \mid R_o,$$

where $\text{Der}(F_o, M) \mid R_o$ is the module of all restrictions to R_o of derivations $F_o \to M$ [**3**]. If M is a trivial $G_S(A)$-module, this reduces to

$$H^2(A; M) \cong \text{Hom} (R, F \circ R \to M, 0)/\text{Hom} (F, M) \mid R,$$

where $\text{Hom}(B, M)$ denotes the set of algebra homomorphisms of a given algebra B into the zero algebra M. If M is also Λ-injective, this reduces further to

$$\begin{aligned} H^2(A; M) &\cong \text{Hom} (R, F \circ R \to M, 0)/\text{Hom} (R, F^2 \cap R \to M, 0) \\ &\cong \text{Hom} (F^2 \cap R, F \circ R \to M, 0) \\ &\cong \text{Hom}_\Lambda (F^2 \cap R/F \circ R, M). \end{aligned}$$

Now consider:

LEMMA 3.2. *If L is any Λ-module and $L^* = \text{Hom}_\Lambda(L, \Gamma)$, then there is a monomorphism $\tau: L \to L^{**}$ given by $\tau(a)(\theta) = \theta(a)$ $[a \in L, \theta \in L^*]$.*

PROOF. If Λ is not a field, Γ is a torsion module. Therefore, if $a \neq 0 \in L$ there always exists a nonzero linear map of Λa into Γ. Since divisible Λ-modules are injective [**1**], this linear map can be extended to a linear map $\theta: L \to \Gamma$ such that $\theta(a) \neq 0$. Hence $a \notin \text{Ker}\, \tau$.

By the lemma there is now a monomorphism

$$\tau: H_2(A; \Lambda) \to \text{Hom}_\Lambda (H^2(A; \Gamma), \Gamma).$$

To see that τ is natural, note that if $A' = F'/R'$ is any presentation of a V-algebra A' as a quotient of a V-free algebra F', then every homomorphism $A \to A'$ can be extended to an extension morphism of $R \to F \to A$ into $R' \to F' \to A'$.

4. **Varieties of groups.** Let $W = W_U$ be the variety of groups determined by a set U of "commutator" identical relations; e.g., the variety of all groups, or all groups of nilpotency class k, or all groups of solvability class k. Then we remark that there exist direct analogues for W of Proposition 2.1 and Examples 2.2 and 2.3, which generalize certain results in [**2**]. (Also there is an analogue of Proposition 3.1, generalizing a result for the variety of all groups [**1**].) These results for W-groups may be proved by essentially the same techniques as before. In particular, to define *W-homology* for a W-group G one uses the "universal envelope" $Z_U(G)$ of G [**5**], and then proceeds similarly to §1. (For the variety of *all* groups, this gives the ordinary homology theory [**1**].)

It would be interesting to know whether or not *every* W-group (or V-algebra) is efficient (cf. [**6**]).

References

1. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N. J., 1956.

2. D. B. A. Epstein, *Finite presentations of groups and 3-manifolds*, Quart. J. Math. Oxford **12** (1961), 205–212.

3. M. Gerstenhaber, *A uniform cohomology theory for algebras*, Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 626–629.

4. J. Knopfmacher, *Universal envelopes for nonassociative algebras*, Quart. J. Math. Oxford **13** (1962), 264–282.

5. ———, *Extensions in varieties of groups and algebras*, Acta Math. **115** (1966), 17–50.

6. R. G. Swan, *Minimal resolutions for finite groups*, Topology **4** (1965), 193–208.

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