A NOTE ON FINITE SOLVABLE K-GROUPS

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A finite K-group is a group having a complemented subgroup lattice i.e. if A is a subgroup of a finite K-group G, there exists a subgroup B such that $G=\{A, B\}$, $A\cap B=1$. The purpose of this note is to identify the structure of a solvable K-group. This reduces to an investigation of a primitive linear solvable K-group which is identified as a semidirect product of a cyclic group of square-free order by a cyclic group of square-free order.

Only *finite* groups will be considered. The notation and the terminology will be that found in the standard references with clarification made whenever necessary.

Nilpotent K-groups are a direct product of elementary Abelian p-groups and the supersolvable K-groups are a subgroup of a direct product of groups of square-free order (e.g. see Suzuki [**5**]). Coupling these results with one of Gaschütz [**1**], a group splits over each normal subgroup whenever all the Sylow p-subgroups are elementary Abelian, and another by F. Gross [**3**], a solvable group is a K-group if and only if the group splits over each normal subgroup, a defining property for a class of solvable K-groups based on the structure of the Sylow p-subgroups is obtained. However this class does not exhaust the set of solvable K-groups as the symmetric group of degree four indicates. So the remaining portion of this note will complete the classification.

Zacher [**6**] has shown the following:

(1) A finite solvable group G is a K-group, if and only if G contains a series of normal subgroups $1=N_0<N_1<\cdots<N_r=G$ such that each N_{i+1}/N_i is a maximal normal nilpotent subgroup of G/N_i, and $\Phi(G/N_i)=1$ for $i=0, 1, \cdots, r-1$.

(2) Each homomorphic image of a solvable K-group is a K-group.

From these the following results of F. Gross [**3**] can be developed:

(3) A solvable group is a K-group if and only if the group splits over each normal subgroup.

(4) Each normal subgroup of a K-group is a K-group.

Note that for (3), if G is a K-group then it splits over each normal subgroup. On the other hand it is enough to note that if G splits over each normal subgroup then $\Phi(G)=1$. Moreover since G is solvable, then a series of subgroups $1=N_0<N_1<\cdots<N_r=G$ exists such that

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N_{i+1}/N_i is the maximal nilpotent normal subgroup of G/N_i. Since for each i, there exists a subgroup M_i such that $G=N_iM_i$, $N_i\cap M_i=1$, then M_i also splits over each normal subgroup. Consequently $\Phi(G/N_i)$ is the identity and from (2) the result follows that G is a K-group.

As for (4) note that by a known result of Gaschütz [2] if a group has the Frattini subgroup the identity element then so does each normal subgroup. So if G is a solvable K-group and N is a normal subgroup of G, then in N a series satisfying (2) exists. Thus N is a K-group.

In general, a solvable group G is a subdirect product of solvable groups H such that H contains precisely one minimal normal subgroup. However if also G is a K-group and A is the kernel of the projection of G onto H, there exists a subgroup B of G such that $G=AB$, $A\cap B=1$, and $B\cong H$ i.e. the direct factors in the subdirect product are isomorphic to subgroups of G. By (2), H is a K-group. Furthermore,

THEOREM 1. *A group G is a solvable K-group if and only if G is a subdirect product of a finite collection of K-groups H, such that each H is isomorphic to a subgroup of G and each H possesses a unique minimal normal subgroup.*

PROOF. The necessity of the conditions follows from the preceding remarks.

Consider then a subdirect product $G\leqq H_1\otimes H_2\otimes \cdots \otimes H_n$, each H_i being a solvable K-group possessing a unique minimal normal subgroup, for which the projections π_i exist, $G\pi_i=H_i$, having kernels A_i, such that $\bigcap_I A_i=1$, for all $i\in I=\{1, \cdots, n\}$. Since $\Phi(H_i)=1$, and $\Phi(G)\pi_i\leqq\Phi(H_i)=1$ for all $i\in I$, then $\Phi(G)\leqq\bigcap_I A_i=1$. Therefore this and the solvability of G implies that G splits over each minimal normal subgroup. So denote by N a normal subgroup of G such that for each G-normal subgroup $N^*<N$, G splits over N^*. Then there exists a subgroup C^* of G such that $N^*C^*=G$, $N^*\cap C^*=1$. Moreover $N=N^*(N\cap C^*)$, N/N^* is a minimal normal subgroup of G/N^*, and thus $N\cap C^*$ is a minimal normal subgroup of C^*. Then note that $H_i=(N^*\pi_i)(C^*\pi_i)$, $(N^*\pi_i)\cap(C^*\pi_i)=1$ for all $i\in I$. However H_i a K-group implies that $C^*\pi_i$ is a K-group and hence $\Phi(C^*\pi_i)=1$. Hence for each $i\in I$, $\Phi(C^*)\leqq A_i$. So $\Phi(C^*)=1$. From this follows the existence of a subgroup B of C^* such that $C^*=(N\cap C^*)B$, $(N\cap C^*)\cap B=1$ and also that $G=N^*(N\cap C^*)B=NB$, $N\cap B=1$. Thus G splits over each normal subgroup. With this and (3) above, G is a K-group.

In order to examine solvable K-groups G having a unique minimal normal subgroup N, it should first be noted that the maximal nilpotent normal subgroup of G, the Fitting subgroup $F(G)$, is N. This follows from $F(G)$ being a direct product of minimal normal subgroups of the K-group G. Furthermore since in a solvable group the centralizer of $F(G)$ is contained in $F(G)$, then $F(G)$ is its own centralizer. Consequently for some prime p, $|F(G)| = p^n$, and there exists a subgroup H such that $G = F(G)H$, $F(G) \cap H = 1$, and H can be considered as an irreducible linear group over a field of p elements acting faithfully on $F(G)$. So let us examine H as an irreducible linear group.

Denote the full linear group on an n-dimensional vector space V^n over a field of p elements by $GL(n, p)$. An irreducible subgroup G of $GL(n, p)$ is *imprimitive* if V^n can be expressed as a direct sum $V_1 \oplus \cdots \oplus V_k$, $k > 1$, of subspaces V_j, that are permuted by the elements of G; otherwise G is *primitive* on V^n. The *systems of imprimitivity*, the V_j, are permuted transitively since G is irreducible. Furthermore by choosing the system of imprimitivity to be a minimum, the subgroup G_i of G that leaves invariant V_i induces on V_i an irreducible primitive subgroup G_i^* of $GL(n/k, p)$. G is then contained in the group $A = BC$, $B \cap C = 1$, B normal in A, B isomorphic to a direct product of k copies of a primitive solvable subgroup of $GL(n/k, p)$, and C isomorphic to a certain solvable transitive subgroup of the symmetric group of degree k. Also each primitive solvable subgroup S of $GL(n, p)$ is contained in a maximal solvable subgroup of $GL(n, p)$ and each subgroup of $GL(n, p)$ containing S is primitive.

THEOREM 2. *A primitive solvable linear group of $GL(n, p)$ is a K-group if and only if it is the semidirect product of a cyclic group of square-free order by a cyclic group of square-free order.*

PROOF. A semidirect product H of a cyclic group of square-free order by a cyclic group of square-free order has elementary Abelian Sylow p-subgroups and hence, by a result of Gaschütz [1], splits over each normal subgroup. By (3) above this is sufficient for G to be a K-group.

On the other hand note that H a solvable K-group implies that $F(H)$ is elementary Abelian and the maximal normal Abelian subgroup of H. Then it follows that $F(H)$ is a subgroup of the multiplicative group of a finite field (see [4]). So $F(H)$ is cyclic and $\Phi(F(H)) = 1$ implies that $F(H)$ is square-free. It is also known (see [4]) that whenever G is a maximal primitive subgroup of $GL(n, p)$ and A is the maximal normal Abelian subgroup of G that G/A is cyclic. Consequently if H is a primitive subgroup, $H \leqq G$, then $AH/A \cong H/A \cap H$

implies that H is a semidirect product of a cyclic group of square-free order be a cyclic group. By (2), $|H/A\cap H|$ is square-free.

COROLLARY 2.1. *A K-group G that is a solvable irreducible subgroup of $GL(n, p)$ is a semidirect product of a subgroup A by a subgroup B such that*

(i) *A is the direct product of k copies of a group H that is the semidirect product of a cyclic group of square-free order by a cyclic group of square-free order and*

(ii) *B is a K-group isomorphic to a solvable transitive subgroup of the symmetric group of degree k.*

PROOF. The semidirect product exists from the earlier remarks and the application of the above theorem to G suffices for the result. The transitive subgroup is determined by the systems of imprimitivity on V^n relative to G.

An alternative representation of a solvable K-group G having a unique minimal normal subgroup N is that G can be considered as a primitive permutation group having $\deg(G)=|N|=p^n$ for some prime p. The results of the previous theorem and corollary can then be applied using the relationship between irreducible subgroups and primitive permutation groups.

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