THE HOMOGENEOUS HILBERT BOUNDARY PROBLEM IN A BANACH ALGEBRA. I

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The homogeneous Hilbert boundary problem [1], [4], [5] had been studied intensively for a long time from different points of view. There is also a list of recent papers primarily concerned with various generalizations of this problem. A typical paper is [6, pp. 436–442].

In the present article we are interested in a solution of the above problem in a complex Banach algebra.

1. **Notations.** The following notation will be used:

$\mathfrak{B}$: a complex Banach algebra with elements $x, y, \cdots$, a unit element e and a norm $\|x\|$ of $x \in \mathfrak{B}$;

$\mathfrak{B}^*$: the adjoint of $\mathfrak{B}$;

$\mathfrak{B}_0$: the subalgebra of $\mathfrak{B}$ defined by $\mathfrak{B}_0 = e\mathfrak{F}$, where $\mathfrak{F}$ is the field of complex numbers;

θ: the zero element of $\mathfrak{B}$;

L: a simple closed smooth positively oriented curve decomposing the complex plane $\mathbf{C}$ into the interior domain D^+ and the exterior domain D^-;

τ, t: the complex coordinates of points on L;

$\sigma(x)$: the spectrum of x;

$R(\lambda; x)$: the resolvent $(\lambda e - x)^{-1}$ of x;

"scroc": a simple closed rectifiable positively oriented curve:

Γ_x: a "scroc" enclosing $\sigma(x)$.

2. **Basic concepts.** An element $x \in \mathfrak{B}$ is said to be *regular* if there is an element x^{-1}, called the inverse of x, such that $xx^{-1} = x^{-1}x = e$.

Let $f(t)$ be a numerically complex-valued function, continuous and nonvanishing on L. *The index* of $f(t)$ on L is the expression

$$\frac{1}{2\pi}\left\{\arg[f(t)]\right\}_L,$$

where $\{\ \}_L$ denotes the total variation (increment) of $\arg[f(t)]$ if t describes L. We shall denote the index of $f(t)$ by $\operatorname{Ind}[f(t)]$.

The exponential and logarithmic functions for elements of a Banach algebra $\mathfrak{B}$ are defined in the sense of [2, pp. 165–173].

Let $\varphi(\tau)$ be a *strongly continuous* function on L to $\mathfrak{B}$. Consider the integral of the Cauchy type

Received by the editors November 8, 1965.

$$\phi(\zeta) = \frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{\tau - \zeta}\, d\tau, \tag{1}$$

where ζ is any point of the plane $\mathbf{C}$. Since for each $x^* \in \mathfrak{B}^*$ the numerically complex-valued function $x^*[\phi(\zeta)]$ is holomorphic (single-valued and differentiable) in the extended plane $\mathbf{C}$ excluding L, the integral (1) defines two holomorphic functions on D^+ and D^- to $\mathfrak{B}$:

$$\phi(\zeta) = \phi^+(\zeta) \text{ if } \zeta \in D^+ \quad \text{and} \quad \phi(\zeta) = \phi^-(\zeta) \text{ if } \zeta \in D^-$$

respectively. In particular, *the function* $\phi(\zeta)$ *is equal to* θ *in the point at infinity* [**3**, p. 165].

The principal value of the singular integral

$$\phi(t) = \frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{\tau - t}\, d\tau, \qquad t \in L,$$

is defined as in the scalar case [**1**, p. 26], [**5**, p. 26], except that the limit is to be understood as a strong limit in $\mathfrak{B}$. It exists if $\varphi(\tau)$ satisfies the Hölder condition on L, i.e. if there are two constants $M>0$ and $0<\alpha\leqq 1$ such that $\|\varphi(t_2)-\varphi(t_1)\| \leqq M|t_2-t_1|^\alpha$ holds for every pair $t_1, t_2 \in L$.

It is well known that the famous Plemelj formulas provide the technique for solving the classical Hilbert boundary problem. In order to extend this problem in the mentioned direction, we turn first to the limiting values of integrals of the Cauchy type in a Banach algebra $\mathfrak{B}$.

THEOREM [**3**]. *Let* $\varphi(t)$ *be a function on L to* $\mathfrak{B}$ *satisfying the Hölder condition on L. If a point* ζ *tends* (*along any path*), *from outside or inside the contour L, to any fixed point* $t \in L$, *then the integral* (1) *converges strongly to the limit*

$$\phi^+(t) = \frac{1}{2}\varphi(t) + \frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{\tau - t}\, d\tau,$$

or

$$\phi^-(t) = -\frac{1}{2}\varphi(t) + \frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{\tau - t}\, d\tau,$$

respectively, where the singular integral is taken in the principal value sense.

In particular, $\phi^+(t) - \phi^-(t) = \varphi(t)$.

Note that $\|\phi^\pm(\zeta) - \phi^\pm(t)\| \to 0$ *uniformly* (with respect to the position t on L) as $\zeta \to t \in L$. This implies therefore the strong continuity of the functions $\phi^{\pm}(t)$ on L.

3. **Formulation of the problem.** By analogy with the scalar case we formulate in a general form the following *Problem* (Hilbert):

Let $f(t)$ be a given function on L to $\mathfrak{B}$ satisfying the Hölder condition and $f(t) \neq \theta$ everywhere on L. Find a function $\phi^+(\zeta)$ holomorphic on D^+ to $\mathfrak{B}$ and a function $\phi^-(\zeta)$ holomorphic on D^- (including the point at infinity) to $\mathfrak{B}$, under the boundary condition

$$\phi^+(t) = f(t)\phi^-(t) \quad on \quad L.$$

We shall solve *a variant of Hilbert's problem* which is formulated as follows.

Let L be a curve in the complex plane $\mathbf{C}$. Let Δ be a *simply-connected* domain of $\mathbf{C}$. Let $\varphi(t, \lambda)$ be a numerically complex-valued function defined for all points $t \in L$ and for all points $\lambda \in \Delta$. One supposes that the function $\varphi(t, \lambda)$ has the following properties:

(1) for each $t \in L$ the function $\varphi(t, \lambda)$ is holomorphic with respect to λ in the domain Δ;

(2) the function $\varphi(t, \lambda)$ satisfies the Hölder condition on L uniformly with respect to λ;

(3) the function $\varphi(t, \lambda)$ is different from zero on the set $L \times \Delta$;

(4) $\operatorname{Ind}[\varphi(t, \lambda)] = 0$ on L.

Let x be a regular element of $\mathfrak{B}$ with $\sigma(x) \subset \Delta$. Define $\varphi(t, x)$ and $\log[\varphi(t, x)]$ as functions on L to $\mathfrak{B}$ by

$$\varphi(t, x) = \frac{1}{2\pi i} \int_{\Gamma_x} R(\lambda; x) \varphi(t, \lambda)\, d\lambda, \tag{2}$$

$$\log[\varphi(t, x)] = \frac{1}{2\pi i} \int_{\Gamma_x} R(\lambda; x) \log[\varphi(t, \lambda)]\, d\lambda, \tag{3}$$

where Γ_x is any "scroc" in Δ.

Find a function $\phi^+(\zeta)$ holomorphic on D^+ to $\mathfrak{B}$ and a function $\phi^-(\zeta)$ holomorphic on D^- (including the point at infinity) to $\mathfrak{B}$ such that

$$\phi^+(t) = \varphi(t, x)\phi^-(t) \tag{4}$$

on L and such that $\phi^-(\infty) \in \mathfrak{B}_0$.

4. **Solution of the problem.** By hypothesis on $\varphi(t, \lambda)$, from (2) and (3) we conclude easily that the function $\varphi(t, x)$ on L to $\mathfrak{B}$ has the following properties:

(1) $\varphi(t, x)$ satisfies the Hölder condition on L;

(2) $\varphi(t, x)$ is a regular element of $\mathfrak{B}$ on L and hence $\varphi(t, x) \neq \theta$ on L;
(3) the function $\log[\varphi(t, x)]$ is single valued on L.

Observe that the nonvanishing function $\varphi(t, \lambda)$ is continuous on the set $L \times \Delta$.

In order to solve the Problem, consider the complex-valued integral of the Cauchy type

$$F(\zeta, \lambda) = \frac{1}{2\pi i} \int_L \frac{\log[\varphi(\tau, \lambda)]}{\tau - \zeta}\, d\tau.$$

The function $F(\zeta, \lambda)$ is holomorphic with respect to ζ in $D^{\pm}$ for each $\lambda \in \Delta$, and for each $\zeta \in D^{\pm}$ it is holomorphic with respect to λ in Δ. Hence, we can define $F(\zeta, x)$ by

$$F(\zeta, x) = \frac{1}{2\pi i} \int_{\Gamma_x} R(\lambda; x) F(\zeta, \lambda)\, d\lambda,$$

from where we deduce easily the abstract integral of the Cauchy type

$$F(\zeta, x) = \frac{1}{2\pi i} \int_L \frac{\log[\varphi(\tau, x)]}{\tau - \zeta}\, d\tau. \tag{5}$$

Since $F^{\pm}(\zeta, \lambda)$ tends to the limit $F^{\pm}(t, \lambda)$ uniformly with respect to the position t on L and with respect to λ [because of the hypothesis (2) on the function $\varphi(t, \lambda)$] as $\zeta \to t$, the function $F^{\pm}(t, \lambda)$ is holomorphic in Δ for each $t \in L$. Hence, we can define $\exp F^{\pm}(t, x)$ and $\exp F^{\pm}(\zeta, x)$ by

$$\exp F^{\pm}(t, x) = \frac{1}{2\pi i} \int_{\Gamma_x} R(\lambda; x) \exp[F^{\pm}(t, \lambda)]\, d\lambda,$$

$$\exp F^{\pm}(\zeta, x) = \frac{1}{2\pi i} \int_{\Gamma_x} R(\lambda; x) \exp[F^{\pm}(\zeta, \lambda)]\, d\lambda.$$

The solution of the Problem is given by

$$\phi^{+}(\zeta) = y \exp F^{+}(\zeta, x),$$
$$\phi^{-}(\zeta) = y \exp F^{-}(\zeta, x),$$

where y is any fixed element of $\mathfrak{B}_0$ and $F^{\pm}(\zeta, x)$ is defined by (5).

To prove this, we start with the product

$$\mathfrak{P}(t, x) \equiv \exp[F^{+}(t, x)] \exp[-F^{-}(t, x)].$$

According to [2, p. 169],

$$\mathfrak{P}(t, x) = \frac{1}{2\pi i} \int_{\Gamma_x} R(\lambda; x) \exp[F^{+}(t, \lambda)] \exp[-F^{-}(t, \lambda)]\, d\lambda$$
$$= \exp[F^{+}(t, x) - F^{-}(t, x)].$$

Applying now Plemelj's formulas to the integral (5), we obtain the desired result:

$$\mathfrak{P}(t, x) = \exp\{\log[\varphi(t, x)]\} = \frac{1}{2\pi i}\int_{\Gamma_x} R(\lambda; x) \exp\{\log[\varphi(t, \lambda)]\} d\lambda$$

$$= \frac{1}{2\pi i}\int_{\Gamma_x} R(\lambda; x)\varphi(t, \lambda)\, d\lambda = \varphi(t, x).$$

The solution is *unique*. Suppose that there exists another solution $\phi_0^\pm(\zeta)$. Since $\phi^\pm(\zeta)$ are regular elements of $\mathfrak{B}$, we may introduce the auxiliary functions

$$G^+(\zeta) = \phi_0^+(\zeta)[\phi^+(\zeta)]^{-1},$$

$$G^-(\zeta) = \phi_0^-(\zeta)[\phi^-(\zeta)]^{-1},$$

holomorphic in D^+ and D^- respectively. By virtue of the boundary condition (4) it follows that $G^+(t)\ [G^-(t)]^{-1} = e$ on L, i.e. $G^+(t) = G^-(t)$. By the theorem of analytic continuation (valid for functions on scalars to $\mathfrak{B}$) there exists a function $G(\zeta)$, holomorphic in the extended plane $\mathbf{C}$, which coincides with $G^+(\zeta)$ in D^+ and with $G^-(\zeta)$ in D^-. According to the theorem of Liouville, $G(\zeta) \equiv \text{const}$. The solution is established.

REMARK. It is easy to verify that the solution of the Problem is independent upon the choice of the holomorphic branch of $\log[\phi(t, \lambda)]$ in Δ.

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