A NOTE ON QUASI-ASSOCIATIVE ALGEBRAS

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By "algebra" we will mean nonassociative algebra with identity over a field Φ of characteristic $\neq 2$. We do not assume finite dimensionality. If $\mathfrak{A} = (\mathfrak{X}, \cdot)$ is an algebra on a vector space $\mathfrak{X}$ and $\lambda \in \Phi$ then the λ-*mutation* of $\mathfrak{A}$ is

$$\mathfrak{A}^{(\lambda)} = (\mathfrak{X}, \cdot_\lambda)$$

where a new multiplication on $\mathfrak{X}$ is introduced by

$$x \cdot_\lambda y = \lambda x \cdot y + (1 - \lambda) y \cdot x. \tag{1}$$

A mutation $\mathfrak{D}^{(\lambda)}$ of an associative algebra for $\lambda \neq 1/2$ is called a *split quasi-associative algebra*. An algebra $\mathfrak{A}$ is called *quasi-associative* if it is a *form* of a split quasi-associative algebra, i.e., there is a *splitting field* $\Omega \supset \Phi$ and $\lambda \in \Omega$, $\lambda \neq 1/2$ such that the extension $\mathfrak{A}_\Omega$ is a λ-mutation of an associative algebra $\mathfrak{D}$ over Ω:

$$\mathfrak{A}_\Omega = \mathfrak{D}^{(\lambda)}. \tag{2}$$

Quasi-associative algebras play an important role in the theory of nonassociative algebras [**1**, pp. 581–584]; [**2**, pp. 192–193]. Although the split quasi-associative algebras have a natural representation, the nonsplit quasi-associative algebras have always seemed to need a similar treatment. It is the purpose of this note to relate them in a natural way to associative algebras, Jordan algebras, Lie algebras, and involutions.

The idea is roughly the following. If $\mathfrak{A}$ is a nonsplit quasi-associative algebra we will find a split quasi-associative algebra $\mathfrak{D}^{(\lambda)}$ (for $\mathfrak{D}$ associative) with an automorphism * such that $\mathfrak{A}$ is the subalgebra of fixed points:

$$\mathfrak{A} = \mathfrak{H}(\mathfrak{D}^{(\lambda)}, *).$$

The automorphism * of $\mathfrak{D}^{(\lambda)}$ will at the same time be an involution on $\mathfrak{D}$. Conversely, if $\mathfrak{D}$ is an associative algebra with an involution * and a suitable element λ in its center then

$$\mathfrak{A} = \mathfrak{H}(\mathfrak{D}^{(\lambda)}, *)$$

will be a nonsplit quasi-associative algebra.

This correspondence between nonsplit quasi-associative algebras

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and associative algebras with involution depends on the scalar λ and the associative algebra $\mathfrak{D}$, and these are not uniquely determined by the representation (2). Indeed, if $\mathfrak{D}^*$ is the opposite algebra (anti-isomorphic) to $\mathfrak{D}$ then

$$\mathfrak{D}^{(\lambda)} = \mathfrak{D}^{*(1-\lambda)}. \tag{3}$$

To obtain a natural one-to-one correspondence we must consider not only nonsplit quasi-associative algebras $\mathfrak{A}$ and associative algebras $\mathfrak{D}$ with involution * but also a choice of λ; what we obtain is a correspondence between pairs $(\mathfrak{A}, \lambda)$ and triples $(\mathfrak{D}, *, \lambda)$.

Instead of λ it will be technically more convenient to work with

$$\Delta = 2\lambda - 1. \tag{4}$$

The square of this element

$$\delta = \Delta^2 \tag{5}$$

is called the *discriminant* of the algebra $\mathfrak{A}$ and is an element of the base field Φ intrinsically determined by $\mathfrak{A}$. Indeed, we have the following interesting characterization of quasi-associative algebras.

THEOREM. *If $\mathfrak{A}$ is a quasi-associative algebra there exists a nonzero element $\delta \in \Phi$ such that the associators $[x, y, z]$ in $\mathfrak{A}$ and $[x, y, z]^+$ in $\mathfrak{A}^+$ are related by*

$$[x, y, z] = (1 - \delta)[x, y, z]^+. \tag{6}$$

Conversely, if $\mathfrak{A}$ is an algebra in which (6) *holds for some $\delta \neq 0$ then $\mathfrak{A}$ is quasi-associative.*

If we agree to set $\delta = 1$ in case $\mathfrak{A}$ itself is associative then δ is uniquely determined by $\mathfrak{A}$, and $\mathfrak{A}$ is a split quasi-associative algebra if and only if $\delta \in \Phi^2$ is a square in Φ.

PROOF. If $\mathfrak{B}$ is an arbitrary algebra and we use (1) to compute the associators $[x, y, z]^{(\lambda)}$ in $\mathfrak{B}^{(\lambda)}$, $[x, y, z]$ in $\mathfrak{B}$, and $[x, y, z]^+$ in $\mathfrak{B}^+$ we obtain the relation

$$\begin{aligned} [x, y, z]^{(\lambda)} = \lambda(2\lambda - 1)[x, y, z] + (1 - \lambda)(2\lambda - 1)[z, y, x] \\ + 4\lambda(1 - \lambda)[x, y, z]^+. \end{aligned} \tag{7}$$

In particular, if $\mathfrak{B} = \mathfrak{D}$ is associative and $\mathfrak{A}$ is quasi-associative as in (2) then (7) reduces to (6) with δ defined by (4) and (5). This also shows that $\delta \in \Phi$ is uniquely determined if $\mathfrak{A}$ is not associative since we can find elements in $\mathfrak{A}$ for which the left side of (6) does not vanish.

Conversely, suppose (6) is satisfied for some $\delta \neq 0$ and let μ be a root of the quadratic equation $4\mu(1-\mu) = 1 - \delta^{-1}$ in an extension field

$\Omega=\Phi(\mu)$. Applying (7) with λ, $\mathfrak{B}$ replaced by μ, $\mathfrak{A}_\Omega$ and using the hypothesis (6) together with the observation that $[z, y, x]^+=-[x, y, z]^+$ by commutativity, we wind up with the formula

$$[x, y, z]^{(\mu)} = \{\mu(2\mu-1)(1-\delta) - (1-\mu)(2\mu-1)(1-\delta) + 4\mu(1-\mu)\}[x, y, z]^+$$

for the associator in $\mathfrak{A}_\Omega^{(\mu)}$. Now by construction

$$\mu(2\mu-1) = \tfrac{1}{2}(2\mu-1+\delta^{-1}),$$
$$(1-\mu)(2\mu-1) = \tfrac{1}{2}(2\mu-1-\delta^{-1}),\ 4\mu(1-\mu) = (1-\delta^{-1});$$

so the term in braces is

$$\delta^{-1}(1-\delta) + (1-\delta^{-1}) = 0$$

and hence $\mathfrak{D}=\mathfrak{A}_\Omega^{(\mu)}$ is associative. The general transitivity relation

$$\{\mathfrak{A}^{(\lambda)}\}^{(\mu)} = \mathfrak{A}^{(\lambda\odot\mu)},\quad \lambda\odot\mu = 1-\lambda-\mu+2\lambda\mu \tag{8}$$

for mutations then implies

$$\mathfrak{A}_\Omega = \mathfrak{A}_\Omega^{(1)} = \mathfrak{A}_\Omega^{(\mu\odot\lambda)} = \{\mathfrak{A}_\Omega^{(\mu)}\}^{(\lambda)} = \mathfrak{D}^{(\lambda)}$$

as in (2) for

$$\lambda = \mu/(2\mu-1) \qquad (\text{note } \delta \neq 0 \Rightarrow \mu \neq \tfrac{1}{2}),$$

and $\mathfrak{A}$ is quasi-associative.

Finally, if $\mathfrak{A}$ is split the discriminant is $1\in\Phi^2$ if $\mathfrak{A}$ is associative and $\Delta^2\in\Phi^2$ by (4) and (5) otherwise. Conversely, if $\delta\in\Phi^2$ then either $\delta=1$, in which case $\mathfrak{A}$ is associative and trivially split, or else Δ and λ are in Φ by (4) and (5). In this case $\mu=\lambda/(2\lambda-1)$ is also in Φ, hence $\Omega=\Phi(\mu)=\Phi$ and by the above $\mathfrak{A}=\mathfrak{A}_\Omega=\mathfrak{D}^{(\lambda)}$ is split.

The actual correspondence mentioned in the introduction is constructed as follows. To a pair $(\mathfrak{A}, \Delta)$ where $\mathfrak{A}$ is a nonsplit quasi-associative algebra and Δ a particular square root of its discriminant δ we associate a triple $(\mathfrak{D}, *, \Delta)$ consisting of an algebra $\mathfrak{D}$, a linear map $*$ of period 2 on $\mathfrak{D}$, and an element Δ in the center of $\mathfrak{D}$. The algebra $\mathfrak{D}$ is defined as the mutation of a certain extension of $\mathfrak{A}$:

$$\mathfrak{D} = \mathfrak{A}_\Omega^{(\mu)} \quad \text{where } \Omega = \Phi(\Delta),\ \mu = \tfrac{1}{2}(1+\Delta^{-1}). \tag{9a}$$

Since Ω is a quadratic field, an automorphism $*$ of $\Omega=\Phi+\Phi\Delta$ is completely determined by taking $\Delta^*=-\Delta$, and this extends to an automorphism $*\otimes 1$ of $\mathfrak{A}_\Omega=\Omega\otimes\mathfrak{A}$:

$$(\omega\otimes a)^* = \omega^*\otimes a,\quad (\alpha+\beta\Delta)^* = \alpha-\beta\Delta \qquad \text{for } \omega\in\Omega,\ a\in\mathfrak{A},\ \alpha, \beta\in\Phi. \tag{9b}$$

The element in the center of $\mathfrak{D}$ is just a scalar multiple of the identity:

$$\Delta = \Delta 1. \tag{9c}$$

In the reverse direction, given an associative algebra $\mathfrak{D}$ with involution $*$ and an element Δ of the center with $\Delta^* = -\Delta$, $\Delta^2 \in \Phi 1$, $\Delta^2 \notin \Phi^2 1$ we associate to the triple $(\mathfrak{D}, *, \Delta)$ a pair $(\mathfrak{A}, \Delta)$ consisting of an algebra $\mathfrak{A}$ and an element Δ in an extension field of Φ. The algebra $\mathfrak{A}$ is defined as the subalgebra of a certain mutation of $\mathfrak{D}$ left fixed by $*$:

$$\mathfrak{A} = \mathfrak{H}(\mathfrak{D}^{(\lambda)}, *) \quad \text{where } \lambda = \tfrac{1}{2}(1 + \Delta). \tag{10a}$$

The assumptions about Δ insure that it belongs to a quadratic extension $\Omega = \Phi(\Delta) \subset \mathfrak{D}$:

$$\Delta 1 = \Delta. \tag{10b}$$

THEOREM. *The correspondence set up by* (9) *and* (10) *is a natural one-to-one correspondence*

$$(\mathfrak{A}, \Delta) \leftrightarrow (\mathfrak{D}, *, \Delta)$$

between the nonsplit quasi-associative algebras $\mathfrak{A}$ with discriminant Δ^2 and the associative algebras $\mathfrak{D}$ with involution $*$ which satisfy $[\mathfrak{D}, [\mathfrak{D}, \mathfrak{D}]] \neq 0$ and which contain an element Δ in their center satisfying $\Delta^* = -\Delta$, $\Delta^2 \in \Phi 1$, $\Delta^2 \notin \Phi^2 1$.

Under this correspondence the Jordan algebra of $*$-symmetric elements of $\mathfrak{D}$ is just the symmetrized algebra of $\mathfrak{A}$

$$\mathfrak{A}^+ = \mathfrak{H}(\mathfrak{D}, *) \tag{11}$$

and the Lie algebra of $*$-skew elements is isomorphic to the skew algebra of $\mathfrak{A}$

$$\mathfrak{A}^- \cong \mathfrak{S}(\mathfrak{D}, *). \tag{12}$$

PROOF. First assume we are given $(\mathfrak{A}, \Delta)$. Then (6) holds for $\delta = \Delta^2$, and in the proof of the first theorem we saw that $\mathfrak{D} = \mathfrak{A}_\Omega^{(\mu)}$ is associative since $4\mu(1-\mu) = (1+\Delta^{-1})(1-\Delta^{-1}) = 1 - \Delta^{-2} = 1 - \delta^{-1}$. Since $*$ is an automorphism of $\mathfrak{A}_\Omega$ and $\mu^* = 1-\mu$ we see that for $x, y \in \mathfrak{D} = \mathfrak{A}_\Omega^{(\mu)}$

$$(xy)^* = (x \cdot_\mu y)^* = x^* \cdot_{\mu^*} y^* = x^* \cdot_{1-\mu} y^* = y^* \cdot_\mu x^* = y^* x^*$$

and $*$ is an involution on $\mathfrak{D}$. Clearly $\Delta 1$ is an element of the center satisfying $\Delta^* = -\Delta$, $\Delta^2 = \delta 1 \in \Phi 1$, and $\Delta^2 \notin \Phi^2 1$ by the first theorem since $\mathfrak{A}$ is nonsplit. Finally, we have the formula

$$4[x, y, z]^+ = [y, [x, z]] \tag{13}$$

in an associative algebra $\mathfrak{D}$. Since $\mathfrak{A}$ is not associative and associators in $\mathfrak{A}^+$ and $\mathfrak{D}^+$ coincide, (6) shows $[\mathfrak{D}, [\mathfrak{D}, \mathfrak{D}]] \neq 0$. Thus the triple $(\mathfrak{D}, *, \Delta)$ defined by (9a), (9b), (9c) is of the required type.

Conversely, suppose we are given $(\mathfrak{D}, *, \Delta)$. Then $\lambda = \frac{1}{2}(1+\Delta)$ has $\lambda^* = 1-\lambda$, so the fact that $*$ is an involution on $\mathfrak{D}$ implies

$$(x \cdot_\lambda y)^* = y^* \cdot_{\lambda^*} x^* = y^* \cdot_{1-\lambda} x^* = x^* \cdot_\lambda y^*$$

and $*$ is an automorphism of $\mathfrak{D}^{(\lambda)}$. Thus $\mathfrak{A} = \mathfrak{H}(\mathfrak{D}^{(\lambda)}, *)$ is an algebra. Since Δ is a skew element of the center we have $\mathfrak{D}^{(\lambda)} = \mathfrak{H}(\mathfrak{D}^{(\lambda)}, *) + \mathfrak{S}(\mathfrak{D}^{(\lambda)}, *) = \mathfrak{A} + \mathfrak{A}\Delta = \mathfrak{A}_\Omega$ and $\Omega = \Phi(\Delta)$ is a splitting field for $\mathfrak{A}$. Since $\delta \neq 1$ and $[\mathfrak{D}, [\mathfrak{D}, \mathfrak{D}]] \neq 0$ we see by (13) and (6) that $\mathfrak{A}$ is not associative, so it is quasi-associative with discriminant δ. By the first theorem, $\delta \notin \Phi^2$ implies $\mathfrak{A}$ is nonsplit. Thus the pair $(\mathfrak{A}, \Delta)$ defined by (10a), (10b) is of the required type.

The correspondence $(\mathfrak{A}, \Delta) \to (\mathfrak{D}, *, \Delta) \to (\mathfrak{A}, \Delta)$ is the identity since the resulting algebra is $\mathfrak{H}(\mathfrak{D}^{(\lambda)}, *)$ where $\mathfrak{D} = \mathfrak{A}_\Omega^{(\mu)}$, so by (8) $\mathfrak{D}^{(\lambda)} = \mathfrak{A}_\Omega$, and hence by (9b) $\mathfrak{H}(\mathfrak{A}_\Omega, *) = \mathfrak{A}$. Going the other way, the correspondence $(\mathfrak{D}, *, \Delta) \to (\mathfrak{A}, \Delta) \to (\mathfrak{D}, *, \Delta)$ is the identity since the resulting algebra is $\mathfrak{A}_\Omega^{(\mu)}$ where we saw that $\mathfrak{A} = \mathfrak{H}(\mathfrak{D}^{(\lambda)}, *)$ has $\mathfrak{A}_\Omega = \mathfrak{D}^{(\lambda)}$ and hence by (8) $\mathfrak{A}_\Omega^{(\mu)} = \mathfrak{D}$, and since the resulting involution on $\mathfrak{A}_\Omega = \Omega \otimes \mathfrak{A}$ agrees with $*$ on $\mathfrak{A}$ (both are the identity) and on Ω $(\Delta^* = -\Delta)$. Thus the correspondence $(\mathfrak{A}, \Delta) \leftrightarrow (\mathfrak{D}, *, \Delta)$ is one-to-one.

To prove (11) we note that both sides consist of the $*$-symmetric elements with the multiplication induced by $\mathfrak{D}^{(\lambda)+} = \mathfrak{D}^+$. To prove (12), we have noted that $\mathfrak{S}(\mathfrak{D}, *) = \mathfrak{A}\Delta$, and the map $a \to a\Delta$ is an isomorphism of $\mathfrak{A}^-$ onto $\mathfrak{A}\Delta$ because of the formula $[x, y] = [x, y]_\mathfrak{D}\Delta$ relating the commutator in $\mathfrak{A}$ with that in $\mathfrak{D}$ by means of (1) and (2).

In closing, we remark that the theorem reduces the classification of simple nonsplit quasi-associative algebras to the classification of simple associative algebras with involutions of the second kind. In fact, the simple nonsplit quasi-associative algebras are precisely the $\mathfrak{H}(\mathfrak{D}^{(\lambda)}, *)$ for simple associative $\mathfrak{D}$ with $[\mathfrak{D}, [\mathfrak{D}\mathfrak{D}]] \neq 0$, $*$ an involution of the second kind, and $\lambda \neq \frac{1}{2}$ an element of the center such that $\lambda + \lambda^* = 1$. Two such algebras $\mathfrak{H}(\mathfrak{D}^{(\lambda)}, *)$ and $\mathfrak{H}(\mathfrak{E}^{(\mu)}, *)$ are isomorphic if and only if either $\mathfrak{D}$ is isomorphic to $\mathfrak{E}$ and $\lambda = \mu$ or $\mathfrak{D}$ is anti-isomorphic to $\mathfrak{E}$ and $\lambda = 1 - \mu$. (These are just the cases suggested by (3).)

References

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