ON THE D-SERIES OF A FINITE GROUP

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Introduction. Let G be a group. The lower central series of G is defined inductively by: $G_1 = G$ and $G_{k+1} = [G_k, G]$ the subgroup of G generated by the set of all commutators $[\sigma, \tau]$ where $\sigma \in G_k$ and $\tau \in G$. By definition G is nilpotent if $G_k = 1$ for some positive integer k.

Let $Z[G]$ denote the integral group ring of G and I the ideal spanned by the elements $\sigma - 1$ where $\sigma \in G$. I is called the augmentation ideal or the fundamental ideal of $Z[G]$ and the powers of I give rise to the descending series of fully invariant subgroups of G,

$$\cdots \subset D_2(G) \subset D_1(G) = G$$

where $D_k(G)$ consists of those elements σ in G such that $\sigma \equiv 1 \pmod{I^k}$. We refer to this as the D-series of G.

For each k, $G_k \subset D_k(G)$ and the map $\Delta: \sigma \to \sigma - 1$ induces a homomorphism Δ_k from G_k/G_{k+1} into I^k/I^{k+1}. The image of Δ_k is the set of homogeneous Lie elements (in the $\Delta\sigma$) in the module I^k/I^{k+1}, and the kernel is $G_k \cap D_{k+1}/G_{k+1}$. (See Cohn [**1**].) Also $G_2 = D_2(G)$ for all groups, and all $G_k = D_k(G)$ if G is a free group. (This last result is due to Magnus [**5**].)

It has been conjectured that all $G_k = D_k(G)$ for arbitrary groups G. We do not prove this, but we do show that for *finite groups* $\bigcap G_k = \bigcap D_k(G)$, or, in other words, that the groups at which the two descending series become constant coincide. This has the immediate corollary that a finite group G is nilpotent if and only if $D_k(G) = 1$ for some positive integer k.

We begin with the following characterization of p-groups among all finite groups. This theorem appears in a paper of Gruenberg [**1**]. We give a simple proof of it here.

THEOREM 1. *Let G be a finite group. G is a p-group for some prime p if and only if $\bigcap I^k = 0$.*

PROOF. Suppose G is a finite p-group for some prime p. The canonical ring homomorphism from $Z[G]$ to $\Gamma[G]$, the group algebra of G over $GF(p)$ the field with p elements, takes I onto the augmentation ideal Δ of $\Gamma[G]$. It is known (Jennings [**4**]) that Δ is precisely the radical of $\Gamma[G]$ and hence $\Delta^l = 0$ for some positive integer l. Consequently $I^l \subset pZ[G]$. Hence $\bigcap I^k \subset \bigcap p^k Z[G]$ which is clearly 0.

Conversely, if $\bigcap I^k = 0$, it follows that $\bigcap D_k(G) = 1$ and hence that $\bigcap G_k = 1$. Since G is finite, this implies that G is nilpotent, hence a direct product of p-groups. To show G is a p-group, we show that if G

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contains commuting elements σ and τ of different prime power orders p and q, then $\bigcap I^k \neq 0$. If $\sigma' = \sigma^{p-1} + \cdots + 1$ and $\tau' = \tau^{q-1} + \cdots + 1$, then $(\sigma - 1)\sigma' = 0$ and $(\tau - 1)\tau' = 0$. Let $ap + bq = 1$, and set $\alpha = a\sigma' + b\tau'$; then $(\sigma - 1)(\tau - 1)\alpha = 0$. Under the augmentation map, α goes into $ap + bq = 1$, whence $\alpha = 1 - \beta$ for some β in I. Therefore $\beta(\sigma - 1)(\tau - 1) = (\sigma - 1)(\tau - 1)$, and since $\beta \in I$ and $(\sigma - 1)(\tau - 1) \in I^2$, it follows that $(\sigma - 1)(\tau - 1) \in I^k$ for all k.

The following result proves useful. We omit the proof which is straightforward.

LEMMA 1. *Let G be a finite group. Then $\bigcap G_k = \bigcap N$ where N runs through all normal subgroups of G of prime power index.*

Let N be a normal subgroup of G. The following standard facts can be found in Fox [2]. The kernel of the canonical ring homomorphism $Z[G] \to Z[G/N]$ is the ideal generated by the set of all $\tau - 1$, $\tau \in N$. If we denote this ideal by $I(N)$, then $Z[G/N] \simeq Z[G]/I(N)$. Also, if $\sigma \in G$ and $\sigma - 1 \in I(N)$, then $\sigma \in N$.

LEMMA 2. *Let N be a normal subgroup of G. N has prime power index in G if and only if $I(N) = \bigcap [I(N) + I^k]$.*

PROOF. If we identify $Z[G/N]$ with $Z[G]/I(N)$, the kth power of the augmentation ideal of $Z[G/N]$ is identified with $[I(N) + I^k]/I(N)$. The result now follows immediately from Theorem 1.

THEOREM 2. *Let G be a finite group. Then $\bigcap G_k = \bigcap D_k(G)$.*

PROOF. Since $G_k \subset D_k(G)$ for all k, the inclusion one way is clear. To show $\bigcap D_k(G) \subset \bigcap G_k$, let $\sigma \in \bigcap D_k(G)$. Then $\sigma - 1 \in \bigcap I^k$, whence for all normal N of prime power index, $\sigma - 1 \in I(N)$. Therefore $\sigma \in N$ for all such N, and $\sigma \in \bigcap G_k$.

COROLLARY 1. *Let G be a finite group. G is nilpotent if and only if $D_k(G) = 1$ for some positive integer k.*

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