ON THE D-SERIES OF A FINITE GROUP

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Introduction. Let \( G \) be a group. The lower central series of \( G \) is defined inductively by: \( G_1 = G \) and \( G_{k+1} = \{G_k, G\} \); the subgroup of \( G \) generated by the set of all commutators \([\sigma, \tau]\) where \( \sigma \in G_k \) and \( \tau \in G \). By definition \( G \) is nilpotent if \( G_k = 1 \) for some positive integer \( k \).

Let \( Z[G] \) denote the integral group ring of \( G \) and \( I \) the ideal spanned by the elements \( \sigma - 1 \) where \( \sigma \in G \). \( I \) is called the augmentation ideal or the fundamental ideal of \( Z[G] \) and the powers of \( I \) give rise to the descending series of fully invariant subgroups of \( G \),

\[ \cdots \subseteq D_2(G) \subseteq D_1(G) = G \]

where \( D_k(G) \) consists of those elements \( \sigma \) in \( G \) such that \( \sigma \equiv 1 \pmod{I^k} \). We refer to this as the \( D \)-series of \( G \).

For each \( k \), \( G_k \subseteq D_k(G) \) and the map \( \Delta: \sigma \mapsto \sigma - 1 \) induces a homomorphism \( \Delta_k \) from \( G_k/G_{k+1} \) into \( I^k/I^{k+1} \). The image of \( \Delta_k \) is the set of homogeneous Lie elements (in the \( \Delta \sigma \)) in the module \( I^k/I^{k+1} \), and the kernel is \( G_k \cap D_{k+1}/G_{k+1} \). (See Cohn [1].) Also \( G_2 = D_2(G) \) for all groups, and all \( G_k = D_k(G) \) if \( G \) is a free group. (This last result is due to Magnus [5].)

It has been conjectured that all \( G_k = D_k(G) \) for arbitrary groups \( G \). We do not prove this, but we do show that for finite groups \( \cap G_k = \cap D_k(G) \), or, in other words, that the groups at which the two descending series become constant coincide. This has the immediate corollary that a finite group \( G \) is nilpotent if and only if \( D_k(G) = 1 \) for some positive integer \( k \).

We begin with the following characterization of \( p \)-groups among all finite groups. This theorem appears in a paper of Gruenberg [1]. We give a simple proof of it here.

**Theorem 1.** Let \( G \) be a finite group. \( G \) is a \( p \)-group for some prime \( p \) if and only if \( \cap I^k = 0 \).

**Proof.** Suppose \( G \) is a finite \( p \)-group for some prime \( p \). The canonical ring homomorphism from \( Z[G] \) to \( \Gamma[G] \), the group algebra of \( G \) over \( GF(p) \) the field with \( p \) elements, takes \( I \) onto the augmentation ideal \( \Delta \) of \( \Gamma[G] \). It is known (Jennings [4]) that \( \Delta \) is precisely the radical of \( \Gamma[G] \) and hence \( \Delta^l = 0 \) for some positive integer \( l \). Consequently \( I^l \subseteq pZ[G] \). Hence \( \cap I^k \subseteq \cap p^kZ[G] \) which is clearly 0.

Conversely, if \( \cap I^k = 0 \), it follows that \( \cap D_k(G) = 1 \) and hence that \( \cap G_k = 1 \). Since \( G \) is finite, this implies that \( G \) is nilpotent, hence a direct product of \( p \)-groups. To show \( G \) is a \( p \)-group, we show that if \( G \)

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contains commuting elements \(\sigma\) and \(\tau\) of different prime power orders \(p\) and \(q\), then \(\cap I^k \neq 0\). If \(\sigma' = \sigma^{p-1} + \cdots + 1\) and \(\tau' = \tau^{q-1} + \cdots + 1\), then \((\sigma - 1)\sigma' = 0\) and \((\tau - 1)\tau' = 0\). Let \(a\sigma + bq = 1\), and set \(\alpha = a\sigma' + b\tau'\); then \((\sigma - 1)(\tau - 1)\alpha = 0\). Under the augmentation map, \(\alpha\) goes into \(a\sigma + bq = 1\), whence \(\alpha = 1 - \beta\) for some \(\beta\) in \(I\). Therefore \(\beta(\sigma - 1)(\tau - 1) = (\sigma - 1)(\tau - 1)\), and since \(\beta \in I\) and \((\sigma - 1)(\tau - 1) \in I^2\), it follows that \((\sigma - 1)(\tau - 1) \in I^k\) for all \(k\).

The following result proves useful. We omit the proof which is straightforward.

**Lemma 1.** Let \(G\) be a finite group. Then \(\cap G_k = \cap N\) where \(N\) runs through all normal subgroups of \(G\) of prime power index.

Let \(N\) be a normal subgroup of \(G\). The following standard facts can be found in Fox [2]. The kernel of the canonical ring homomorphism \(Z[G] \to Z[G/N]\) is the ideal generated by the set of all \(\tau - 1, \tau \in N\). If we denote this ideal by \(I(N)\), then \(Z[G/N] \cong Z[G]/I(N)\). Also, if \(\sigma \in G\) and \(\sigma - 1 \in I(N)\), then \(\sigma \in N\).

**Lemma 2.** Let \(N\) be a normal subgroup of \(G\). \(N\) has prime power index in \(G\) if and only if \(I(N) = \cap [I(N) + I^k]\).

**Proof.** If we identify \(Z[G/N]\) with \(Z[G]/I(N)\), the \(k\)th power of the augmentation ideal of \(Z[G/N]\) is identified with \([I(N) + I^k]/I(N)\). The result now follows immediately from Theorem 1.

**Theorem 2.** Let \(G\) be a finite group. Then \(\cap G_k = \cap D_k(G)\).

**Proof.** Since \(G_k \subseteq D_k(G)\) for all \(k\), the inclusion one way is clear. To show \(\cap D_k(G) \subseteq \cap G_k\), let \(\sigma \in \cap D_k(G)\). Then \(\sigma - 1 \in \cap I^k\), whence for all normal \(N\) of prime power index, \(\sigma - 1 \in I(N)\). Therefore \(\sigma \in N\) for all such \(N\), and \(\sigma \in \cap G_k\).

**Corollary 1.** Let \(G\) be a finite group. \(G\) is nilpotent if and only if \(D_k(G) = 1\) for some positive integer \(k\).

**References**