SOME TOPOLOGICAL PROPERTIES OF THE FUNCTION $n(y)$[1]

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1. **Introduction.** For a continuous function $f(x)$ on the unit interval, let $n(y)$ be the function[2] which counts the number of solutions of the equation $f(x)=y$—i.e., the number of intersections of the graph of $f(x)$ with the line $y_1=y$. The function $n(y)$ was introduced by Banach [**1**], who proved that $f(x)$ is of bounded variation if and only if $n(y)$ is Lebesgue integrable, and derived the formula

$$V(f) = \int n(y)dy \tag{1}$$

for the total variation of $f(x)$. In particular, if $f(x)$ is of bounded variation it has an a.e. finite $n(y)$, but in general we could have $n(y)=\infty$ identically in the range of $f(x)$ (see e.g. §3). The condition T_1 that $n(y)<\infty$ a.e. has many real variable characterizations ([**2**], [**4**, pp. 278–287]); for example, it is equivalent to certain differentiability conditions on $f(x)$. Nina Bary [**2**] has also shown that any continuous function on the unit interval can be written as the sum of three continuous functions satisfying T_1. The paper [**3**] is in a slightly different vein, in which an a.e. equivalent definition of $n(y)$ is given with applications in Fourier series. For example (assuming $f(0)=f(1)$), the condition

$$\int \log^+ n(y)dy < \infty$$

is shown to imply the uniform convergence of the Fourier series of $f(x)$.

The results quoted above are all applications of measure-theoretic arguments to the function $n(y)$; the purpose of this paper is to look at implications of the topological properties of $n(y)$. For example, assume $f(x)$ is a continuous function on the unit interval with the property that $n(y)<\infty$ for every y. Then, it follows from Theorem 1 that $f(x)$ is of bounded variation on a subinterval of any given inter-

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[2] See §2 for a more precise definition.

val $(a, b)\subseteq[0, 1]$. This is a much stronger result than would be possible if just $n(y)<\infty$ a.e. Also, this property is preserved under finite sums; hence, the result of Nina Bary is no longer true if T_1 is strengthened to $n(y)<\infty$, every y. We also remark that if $n(y)$ is everywhere finite, the Fourier series of $f(x)$ then converges uniformly at each point of an open dense set in the unit interval; this, too, is false if we merely assume $n(y)<\infty$ a.e.

§3 is devoted to some results of a negative nature. For example, it is shown by the method of category that there exist continuous functions $f(x)$ such that $n(y)=\infty$ identically for $\min f(x)<y<\max f(x)$. In fact, the function $f(x)$ can even be chosen to satisfy a Hölder condition for an arbitrary exponent $\beta<1$. Thus, for a general y_0 in the range, $f(x)$ oscillates through the line $y=y_0$ infinitely often; we also show that the amplitudes of these oscillations cannot die out too quickly. More precisely, let $y<\alpha$ be a pair of real numbers, and $0<x_1<x_2<\cdots<x_N<1$ a partition of $[0, 1]$. Assume further that $f(x_k)=y$ for each k, and that $f(x)=\alpha$ for at least one x in each of the $N-1$ intervals $\{(x_k, x_{k+1})\}$. We now define $n(y, \alpha)$ as the maximum N for such partitions.[3] Thus $n(y, \alpha)<\infty$ for every $f(x)$ and $y<\alpha$, and $0\leqq n(y, \alpha)\leqq n(y)$; indeed, in the absence of local maxima at $f(x)=y$, $0\leqq n(y, \alpha)\uparrow(1/2)n(y)-\epsilon$ as $\alpha\rightarrow y$, where $0\leqq\epsilon\leqq 1$. It then follows from Theorem 3 that for the general β-Hölder continuous function $f(x)$, where $0<\beta<1/2$, $n(y, \alpha)$ increases sufficiently rapidly as $\alpha\rightarrow y$ so that

$$\int_y^B n(y, \alpha)d\alpha = \infty \text{ for every } y, \qquad A < y < B,$$

where $A=\min f(x)$ and $B=\max f(x)$.

2. **A definition and Theorem 1.** For a continuous function $f(x)\in C[0, 1]$ and a real number y, consider the set $\{x: f(x)=y\}$. If it is of cardinality N, or is the union of N possibly-degenerate closed intervals, we define $n(y, f)=N$; otherwise, $n(y, f)=\infty$. If the function $f(x)$ is understood, we will shorten $n(y, f)$ to $n(y)$. To avoid certain inconveniences, we will also decrease the value of $n(y)$ by one for $y=f(0)$ or $y=f(1)$; for periodic $f(x)$ this is equivalent to assuming that $f(x)$ is defined on the unit circle. We now prove an important continuity property of $n(y)$, which is slightly weaker than the classical condition of lower semicontinuity. While $n(y)$ in general is not lsc, most of the properties that follow from lsc are implied by (2), although perhaps in a weaker form. For example, the set $\{y: n(y)\leqq N\}$

[3] The function $n(y, \alpha)$ was introduced (essentially) in [3]; indeed, if $N(E_{y,\alpha})$ is defined as in [3, p. 591], then $n(y, \alpha+)\leqq N(E_{y,\alpha})+1\leqq n(y, \alpha)+1$.

can differ from a closed set by at most a countable set; see also Corollary 2.3.

LEMMA 2.1. *Assume* $b_k - a_k \to 0$ *and* $a_k < y < b_k$, *where* y *is some real number and* $f(x) \in C[0, 1]$. *Then,*

$$n(y) \leqq \liminf_{k\to\infty} \max \{n(a_k), n(b_k)\}. \tag{2}$$

PROOF. First, assume $0 < x_1 < x_2 < \cdots < x_N < 1$ are the N solutions of $f(x) = y$, where $n(y) = N < \infty$. Choose $\epsilon < (1/4) \min (x_{k+1} - x_k)$, and $\delta > 0$ such that $|f(x) - y| \geqq \delta$ for at least one value of x in each of the $2N$ intervals $(x_k - \epsilon, x_k)$, $(x_k, x_k + \epsilon)$. Thus, if $0 < \delta_1, \delta_2 < \delta$, $2N \leqq n(y - \delta_1) + n(y + \delta_2)$ by the Intermediate Value Theorem, and for sufficiently large k

$$n(y) \leqq \frac{n(a_k) + n(b_k)}{2} \leqq \max\{n(a_k), n(b_k)\}.$$

The case where there are intervals of solutions of $f(x) = y$, or where $y = f(0)$ or $y = f(1)$, can be handled similarly. Likewise, the same argument shows that if $n(y) = \infty$, the right-hand side of (2) cannot be bounded.

REMARK. Similarly, if y is not the ordinate of a local minimum or maximum of $f(x)$ and $y_k \to y$, then $n(y) \leqq \liminf_{k\to\infty} n(y_k)$.

LEMMA 2.2. *Given a nonconstant* $f(x) \in C[0, 1]$, *assume that* $n(y, f) < \infty$ *for all* y *in a set of the second (Baire) category in the range of* $f(x)$. *Then, there exists a nonempty open interval* $(a, b) \subseteq [0, 1]$ *on which* $f(x)$ *is of bounded variation.*

PROOF. Set $E_N = \{y: n(y) \leqq N\}$ and $A = \min f(x)$, $B = \max f(x)$. By hypothesis, $\bigcup E_N$ is a set of the second Baire category in the interval $[A, B]$; hence some set E_N is dense in some nonempty open interval $(\alpha, \beta) \subseteq [A, B]$. It then follows from Lemma 2.1 that $n(y) \leqq N$ for all y, $\alpha < y < \beta$. Let (a, b) be a component interval of $f^{-1}\{(\alpha, \beta)\}$; I claim that the variation of $f(x)$ on (a, b) is bounded by $N(\beta - \alpha)$. (See also formula (1).) This follows from the fact that if $\sum |f(x_{k+1}) - f(x_k)| > N(\beta - \alpha)$ for some partition $a = x_0 < x_1 < x_2 < \cdots < x_M = b$, then some open interval $(\alpha', \beta') \subset [\alpha, \beta]$ must be covered by more than N of the closed intervals $\{f([x_k, x_{k+1}])\}$.

COROLLARY 2.3. *Assume that* $n(y) < \infty$ *for every* y, *except perhaps on a set of the first (Baire) category. Then,* $n(y) = \infty$ *only on a nowhere dense set.*

PROOF. Assume $A < \alpha < \beta < B$, where A, B, E_N are as before. Since

$\bigcup E_N$ is of the second category in $[\alpha, \beta]$, some set E_N is dense in some interval $(\alpha', \beta') \subseteq [\alpha, \beta]$, and $n(y) \leqq N$ for $\alpha' < y < \beta'$ as before. Hence $\{y: n(y) = \infty\}$ is not dense in (α, β).

THEOREM 1. *Assume $n(y, f) < \infty$ except for a countable number of y, or alternately except for a nowhere dense set. Then, there exists a sequence of intervals $(a_n, b_n) \subseteq [0, 1]$, whose union is dense in $[0, 1]$, such that $f(x)$ is of bounded variation in each interval (a_n, b_n).*

PROOF. I claim that every nonempty interval $(c, d) \subseteq [0, 1]$ contains a nonempty interval (a, b) on which $f(x)$ is of bounded variation; this follows by applying Lemma 2.2 to the function $g(x) = f(c + (d-c)x)$. Hence if $\mathfrak{F}$ is the collection of all intervals (r_1, r_2) with rational endpoints and with $f(x)$ of bounded variation on (r_1, r_2), then $\bigcup \mathfrak{F}$ is dense in $[0, 1]$.

REMARK. If $f(x) = 2x + x \sin 1/x$, then $f(x)$ is of bounded variation on each interval $(1/n, 1]$ but not on $(0, 1]$; thus Theorem 1 cannot be improved even if $n(y, f) < \infty$ for every y.

3. **Negative results.** We recall that a subset R of a complete metric space E is called *residual* if it is the complement of a set of the first category in E; i.e., the complement of a countable union of nowhere dense sets. In particular, a residual set would be nonempty (and even dense in the space), and a countable intersection of residual sets is also residual. In the following, let $C[0, 1]$ be the metric space of all continuous functions on $[0, 1]$, with the norm $\|f\| = \max_x |f(x)|$.

THEOREM 2. *For all $f(x)$ in a certain residual set in $C[0, 1]$, $n(y, f) = \infty$ for every y, $\min_x f(x) < y < \max_x f(x)$.*

PROOF. For $f(x) \in C[0, 1]$ and $A = \min f(x)$, $B = \max f(x)$, define

$$K_N = \left\{ f:\ B - A \geqq 2/N,\ \inf_{A+1/N \leqq y \leqq B-1/N} n(y, f) \leqq N \right\}. \tag{3}$$

I claim that each set K_N is nowhere dense in $C[0, 1]$, or, alternately, that any neighborhood of any function $f(x) \in C^1[0, 1]$ contains an open set which is totally disjoint from K_N. Choose $f(x) \in C^1[0, 1]$ and $\eta > 0$, and an integer k such that $3C/k < \eta$, $3C/k < 1/N$, where $C = \max_x |f'(x)|$. Thus $|f(l/k) - f((l-1)/k)| \leqq C/k$ for $1 \leqq l \leqq k$; now define

$$\begin{aligned} f_1(x) &= f_{l-1} + 2C/k \sin 4\pi N k x, \\ &\qquad\qquad (l-1)/k \leqq x \leqq (2l-1)/2k, \\ &= f_{l-1} + 2k(f_l - f_{l-1})(x - (2l-1)/2k), \\ &\qquad\qquad (2l-1)/2k \leqq x \leqq l/k, \end{aligned} \tag{4}$$

where $f_l = f(l/k)$. Hence, $\|f - f_1\| \leqq 3C/k < \eta$, where $\|f\| = \max_x |f(x)|$ as above, and $n(y, f_1) \geqq 2N$ for $\min f_1 < y < \max f_1$. Now, assume $g(x) \in C[0, 1]$, $\|f_1 - g\| < \epsilon = C/4k$. Thus, $f((l-1)/k) \leqq y \leqq f(l/k)$ (or $f_l \leqq y \leqq f_{l-1}$), some l, for every y in the interval $\min g(x) + 2C/k + \epsilon \leqq y \leqq \max g(x) - 2C/k - \epsilon$; and $n(y, g) \geqq 2N - 1$ for these y by the Intermediate Value Theorem. Hence K_N is nowhere dense for any N; the residual set required is the complement of $\bigcup K_N$ plus the constant functions.

The same arguments also apply in the Hölder spaces $C_0^\beta[0, 1]$, $0 < \beta < 1$. Here $C_0^\beta[0, 1]$ is the closure of $C^1[0, 1]$ with respect to the norm $\|f\|_\beta = \|f\| + \sup_{x,y} |f(x) - f(y)| / |x - y|^\beta$; i.e., the set of all $f(x) \in C[0, 1]$ such that $f(x) - f(y) = o(|x - y|^\beta)$ uniformly in x and y.[4]

COROLLARY 3.1. *Assume* $0 < \beta < 1$. *Then, the set of* $f(x) \in C_0^\beta[0, 1]$ *such that* $n(y, f) = \infty$ *for every* y, $\min f(x) < y < \max f(x)$, *is a residual set in* $C_0^\beta[0, 1]$.

PROOF. Define $K_N \subseteq C_0^\beta$ as in (3), and note that

$$\|f - f_1\|_\beta = O(N^\beta / k^{1-\beta}).$$

Alternately, since $0 \leqq n(y, \alpha) \leqq n(y)$, where $n(y, \alpha)$ is as defined in the introduction, Corollary 3.1 could be obtained from

THEOREM 3. *Assume* $0 < \beta < 1$ *and* $\lambda > \beta/(1 - \beta)$. *Then, for every* $f(x)$ *in a certain residual set in* $C_0^\beta[0, 1]$, *and* A, B *as before,*

$$\int_y^B n(y, \alpha)^\lambda d\alpha = \infty \text{ for all } y, \qquad A < y < B. \tag{5}$$

PROOF. For $f(x) \in C_0^\beta[0, 1]$ and $N \geqq 1$, define

$$K_N = \left\{ f\colon B - A \geqq 2/N, \inf_{A + 1/N \leqq y \leqq B - 1/N} \int_y^B n(y, \alpha)^\lambda d\alpha \leqq N \right\}.$$

As before, choose $f(x) \in C^1[0, 1]$, $\eta > 0$, $\delta = 1 - \beta - \beta/\lambda > 0$, $C > \max_x |f'(x)|$, and an integer k such that $3C/k < 1/N$, $50CN/k^\delta < \eta$. Define $f_1(x)$ as in (4), except with $M = [(Nk)^{1/\lambda}]$ replacing N; thus $\|f - f_1\|_\beta = O(M^\beta / k^{1-\beta}) < \eta$. If now $\|f_1 - g\| < \epsilon = C/4k$, then $g(x)$ satisfies $n(y, y + \epsilon) \geqq M - 1$ for $\min g(x) + 1/N \leqq y \leqq \max g(x) - 1/N$, and $\int_y^\beta n(y, \alpha)^\lambda d\alpha \geqq \epsilon(M - 1)^\lambda > N$ for these y and $C > C(\lambda)$. Hence, K_N is nowhere dense in $C_0^\beta[0, 1]$, as before.

REMARKS. Assume $A \leqq y < \alpha \leqq B$ and $f(x) \in C_0^\beta[0, 1]$; then $f(x_1) = y$

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[4] Indeed, if $f(x) - f(y) = o(|x - y|^\beta)$ and $f(0) = f(1)$, then $\|s_n - f\|_\beta \to 0$, where $\{s_n(x)\}$ are the $(C, 1)$ sums of the trigonometric Fourier series of $f(x)$.

and $f(x_2)=\alpha$ for some x_1, x_2, and $\alpha-y\leqq\|f\|_\beta(1/n(y,\alpha))^\beta$ by the definition of $n(y, \alpha)$. Hence,

$$n(y, \alpha) = O(1/(\alpha - y)^{1/\beta})$$

uniformly in y and α, for any $f(x)\in C_0^\beta[0, 1]$. In particular, the integral (5) is always finite for any $\lambda<\beta$. On the other hand, for any $l<1/\beta-1$, it follows from Theorem 3 that $n(y, \alpha)\neq O(1/(\alpha-y)^l)$ for *any* value of y in the interior of the range of a general $f(x)\in C_0^\beta[0, 1]$. As an intermediate result, it can be shown that $n(y, \alpha)\neq O(1/(\alpha-y)^{1/\beta-\epsilon})$ for any $\epsilon>0$ and all y in a residual set in the range, again for the general $f(x)\in C_0^\beta[0, 1]$.

Actually, there is a close relationship between the behavior of $n(y, \alpha)$ and the modulus of continuity for a function $f(x)\in C[0, 1]$, and most of the above results have analogs for more general Banach spaces of continuous functions. For example, let $w(\theta)=\theta\log 1/\theta$, and let $C_0^w[0, 1]$ be the closure of $C^1[0, 1]$ with respect to the norm $\|f\|_w=\max_x|f(x)|+\sup_{x,y}|f(x)-f(y)|/w(|x-y|)$. As before (see footnote 4), $C_0^w[0, 1]$ is identical with the set of $f(x)\in C[0, 1]$ such that $f(x)-f(y)=o(w(|x-y|))$ uniformly in x and y. An easy adaptation of the proof of Theorem 3 now implies that for all $f(x)$ in a residual set in $C_0^w[0, 1]$,

$$\int_y^B \exp n(y, \alpha)^2 d\alpha = \infty \text{ for all } y, \qquad A < y < B,$$

where $A=\min f(x)$, $B=\max f(x)$ as before. In particular, this gives a class of functions $f(x)\in C[0, 1]$ with $\|f\|_\beta=O(1/(1-\beta))$, $0<\beta<1$, such that $n(y)=\infty$ identically for $A<y<B$.

Indeed, the existence of $f(x)\in C[0, 1]$ with $n(y)=\infty$ for $A\leqq y\leqq B$ is part of the folk lore of the subject; see e.g. the result of Kunen in [3, p. 601].

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