q-DIFFERENTIAL EQUATIONS WITH POLYNOMIAL SOLUTIONS

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1. **Introduction.** Some years back T. W. Chaundy [1] discussed the problem of obtaining polynomial solutions of second-order linear differential equations. The object of this paper is to discuss the corresponding problem in the case of q-differential equations. In particular, general types of q-differential equations which yield polynomial solutions of restricted degrees and all degrees have been deduced.

We consider here the problem of finding systems of linear q-differential equations

$$(1) \qquad (F - \lambda G) Y = 0, \qquad \lambda \neq 0, \ \infty,$$

such that when $\lambda = \lambda_n$, the equation is satisfied by a polynomial of degree n (n, a positive integer or zero). Equation (1) is, in general, of the form

$$(\alpha_0 \Delta^p + \alpha_1 \Delta^{p-1} + \cdots + \alpha_p) Y = \lambda(\beta_0 \Delta^p + \beta_1 \Delta^{p-1} + \cdots + \beta_p)$$

where Δ is the operator $(q^{xd/dx} - 1)/x(q-1)$. Also $\alpha_0, \beta_0, \alpha_1, \beta_1, \cdots, \alpha_p, \beta_p$ are $2p+2$ functions of the independent variable x. To determine their ratios we should have $2p+1$ equations involving these functions. We consider the $2p+1$ polynomial solutions of the q-differential equation (1) for different values of λ, so that the ratios are determined; for convenience, writing the fundamental operator Δ as $[\theta]$, where $[\theta] = x\Delta$ (which still leaves the coefficients polynomials in x), and arranging (1) in powers of x, it becomes of the form

$$(2) \qquad F(x, [\theta]) Y = \lambda G(x, [\theta]) Y,$$

where

$$F(x, [\theta]) = \sum_{r=s}^{p} x^r f_r([\theta]), \qquad G(x, [\theta]) = \sum_{r=s'}^{p'} x^r g_r([\theta]),$$

($p \geqq s$, $p' \geqq s'$). In particular, we consider the equation

$$(3) \qquad \left[\sum_{r=s}^{p} x^r f_r([\theta]) \right] Y_n = \lambda_n \left[\sum_{r=s'}^{p'} x^r g_r([\theta]) \right] Y_n,$$

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where Y_n is a polynomial solution of the form

(4) $$Y_n = x^n + a_{n,n-1}x^{n-1} + \cdots + a_{n,0}.$$

By "a polynomial of degree n" we mean here not a polynomial of degree not exceeding n, but "of degree exactly n" thus x^n is always present in Y_n.

Putting (4) in (3) and operating $[\theta]$, we get the identity

$$[x^{n+p}f_p([n]) + \cdots + x^s a_{n,0}f_s([0])]$$
$$\equiv \lambda_n[x^{n+p'}g_{p'}([n]) + \cdots + x^{s'}a_{n,0}g_{s'}([0])].$$

The highest powers of x on the left and right are then $x^{n+p}f_p([n])$ and $x^{n+p'}g_{p'}([n])$, respectively, where $[n]=(q^n-1)/(q-1)$. Thus, unless $p=p'$, either $f_p([n])$ or $g_{p'}([n])$ would remain as isolated terms in (3), and we should need $f_p([n])=0$ or $g_{p'}([n])=0$ for all n, which is impossible. Hence $p=p'$. At the other end of summation, since (3), after operation, is an identity between polynomials of degree $n+p$, it must have the same lowest power of x and therefore there is no loss of generality in assuming $s=s'$. It is, therefore, sufficient to consider (3) in the form

(5) $$\left[\sum_{r=0}^{p} x^r f_r([\theta])\right] Y_n = \lambda_n\left[\sum_{r=0}^{p} x^r g_r([\theta])\right] Y_n.$$

Considering the sequence of polynomials $\{Y_n\}$, we can successively choose $c_{n,n-1}, c_{n,n-2}, \cdots, c_{n,0}$ such that

(6) $$x^n = Y_n + c_{n,n-1}Y_{n-1} + \cdots + c_{n,0}Y_0.$$

Then from (5),

(7) $$F(x, [\theta])x^n = G(x, [\theta])Z_n$$

where

$$Z_n = \lambda_n Y_n + \lambda_{n-1}c_{n,n-1}Y_{n-1} + \cdots + \lambda_0 c_{n,0}Y_0.$$

Obviously, we can suppose that

(8) $$Z_n = \lambda_n x^n + b_{n,n-1}x^{n-1} + \cdots + b_{n,0}.$$

2. Now we proceed to deduce a sufficient form of a q-differential equation which has polynomial solutions of every degree provided that λ_n $(n=0, 1, 2, \cdots)$ are all different.

THEOREM. *The system of equations*

$$\left(\sum_{r=0}^{p} x^{p-r}[\theta][\theta-1]\cdots[\theta-r+1]h_{p-r}([\theta])\right)Y \tag{9}$$
$$= \lambda\left(\sum_{r=0}^{p} x^{p-r}[\theta][\theta-1]\cdots[\theta-r+1]j_{p-r}([\theta])\right)Y$$

(where $h_t([\theta])$ and $j_t([\theta])$ denote polynomials in $[\theta]$ with constant coefficients and $h_0([\theta])$ and $j_0([\theta])\neq 0$) has polynomial solutions of every degree with

$$\lambda_n = h_p([n])/j_p([n])$$

provided that these λ_n ($n=0, 1, 2, \cdots$) are all distinct.

PROOF. Let us assume that for $\lambda=\lambda_n$, equation (9) has polynomial solution

$$Y_n = x^n + a_{n,n-1}x^{n-1} + \cdots + a_{n,0}.$$

Now, the operator $[\theta][\theta-1]\cdots[\theta-r+1]$ appearing in the coefficient of x^{p-r} on either side removes from Y_n powers of x below x^r, for

$$\{[\theta][\theta-1]\cdots[\theta-r+1]\}\sum_{s=0}^{r-1}\alpha_s x^s = 0;$$

so x^p is the lowest power of x that emerges. Also the highest power is x^{n+p} and thus, comparing coefficients of x on both sides in (9), gives $n+1$ equations to determine the $n+1$ constants $a_{n,n-1}, a_{n,n-2}, \cdots, a_{n,0}$ and λ_n. The equations are

$$0 = h_p([n]) - \lambda_n j_p([n]),$$
$$0 = a_{n,n-1}\{h_p([n-1]) - \lambda_n j_p([n-1])\} + \lambda[n]\{h_{p-1}([n]) - \lambda_n j_{p-1}([n])\}$$

and so on. The first gives $\lambda_n = h_p([n])/j_p([n])$ and from the others $a_{n,n-1}, a_{n,n-2}, \cdots, a_{n,0}$ can be determined in succession (without infinity), provided that none of $h_p([r])-\lambda_n j_p([r])$ is zero, which is covered by the condition that λ_n ($n=0, 1, 2, \cdots$) are all different.

3. Next, we examine the generality of the above form given in (9). Writing in short

$$[H(x, [\theta]) - \lambda J(x, [\theta])]Y = 0, \tag{9a}$$

it can be rewritten as

$$[H(x, [\theta-m]) - \lambda J(x, [\theta-m])x^m]\, Y = 0,$$

and, consequently, when m is positive integer, the system

$$(10)\qquad [H(x, [\theta - m]) - \lambda J(x, [\theta - m])]\overline{Y} = 0$$

has polynomial solutions $\overline{Y} = x^m Y_n$ of all degrees m and upwards, with the same set of values of λ. However, the system (10) may have polynomial solutions of degree less than m only by imposing further finite sets of conditions on (10). Then it will become the desired system giving polynomial solutions of all orders. A system of the type (10) becomes, for example,

$$(11)\quad [H(x, [\theta - m]) - \lambda J(x, [\theta - m])][\theta][\theta - 1] \cdots [\theta - m + 1]Y = 0$$

as the operator $[\theta][\theta-1] \cdots [\theta-m+1]$ annihilates all polynomials of degree less than m. So it is of the form (10). To deduce the set of conditions to be imposed on (10) so that it can give the remaining polynomial solutions of degree less than m also, let us write F, G in the form

$$(12)\qquad F(x, [\theta]) = \sum_{r=0}^{p} x^{p-r}\left\{P_r([\theta]) + \sum_{s=0}^{m-1} \frac{\lambda_s e_{r,s}}{[\theta - s]}\right\} \cdot [\theta][\theta - 1] \cdots [\theta - m - r + 1],$$

$$(13)\qquad G(x, [\theta]) = \sum_{r=0}^{p} x^{p-r}\left\{Q_r([\theta]) + \sum_{s=0}^{m-1} \frac{e_{r,s}}{[\theta - s]}\right\} \cdot [\theta][\theta - 1] \cdots [\theta - m - r + 1]$$

where the $e_{r,s}$ are a set of arbitrary constants, and P_r, Q_r are arbitrary polynomials with constant coefficients so as to contribute the $[\theta - m]$ type of factors in (11). Hence, the coefficient of x^{p-r} in each operator is of the form $[\theta-m] \cdots [\theta-m-r+1]$ multiplied by an operator polynomial in $[\theta]$. Thus, (12) and (13) give

$$F(x, [\theta]) - \lambda G(x, [\theta])$$

of the required form $H(x, [\theta-m])-\lambda J(x, [\theta-m])$.

I shall prove now that

$$(14)\qquad F(x, [\theta])Y = \lambda G(x, [\theta])Y$$

has polynomial solutions of degree less than m as simple powers of x with $F(x, [\theta])$, $G(x, [\theta])$ as given in (12), (13). Operating on x^n ($n < m$), (14) gives

$$x^n \sum_{r=0}^{p} x^{p-r}(\lambda_n - \lambda)[n][n - 1] \cdots [1][-1] \cdots [1 - r]$$

which vanishes for $\lambda = \lambda_n$.

Thus, $F(x, [\theta])-\lambda_n G(x, [\theta])$ annihilates x^n. Hence (14) for $\lambda=\lambda_n$ has polynomial solutions of degree less than m as simple powers of x.

Thus, the form (14) with $F(x, [\theta])$, $G(x, [\theta])$, as defined in (12), (13), has polynomial solutions of all degrees m upward, and the polynomials of degree less than m are simple powers of x.

4. **The type $\mathfrak{A}$ and $\mathfrak{A}_m$.** We shall distinguish the forms given in (9) as type $\mathfrak{A}$ and those defined in (12), (13) as type $\mathfrak{A}_m$; $\mathfrak{A}$ is in fact $\mathfrak{A}_0$. (For $m=0$, the series $\sum_{s=0}^{m-1}$ in (12) and (13) do not appear.)

The q-differential equations (9) and (14) (with forms of F and G as in (12) and (13)) are both of rank "p" in the sense that there are $p+1$ powers of x and also p distinct steps between these powers in the q-differential equation. However, in (9) (an equation of the type $\mathfrak{A}$) the least order of $[\theta]$ is at least p, since the coefficient of the absolute term (the term independent of x) contains the factor $[\theta][\theta-1] \cdots [\theta-p+1]$. In (14) (an equation of the type $\mathfrak{A}_m$) however, the least order of $[\theta]$ is at least $m+p-1$. Therefore, the order of (9) is at least p, and that of (14) is at least $m+p-1$; $m\geqq 1$.

Conversely, then, the equation of a given order n of types $\mathfrak{A}$, $\mathfrak{A}_1$ are of rank at most n; the rank of any type $\mathfrak{A}_m$ is at most $n-m+1$. Thus, the second-order equations of these types are not necessarily of q-hypergeometric type but may be of rank two.

As a simple illustration, consider the possible first-order q-differential equation. The type $\mathfrak{A}$ then gives most generally

$$[x\{a[\theta] - b\} - c[\theta]]Y = \lambda[x\{a'[\theta] - b'\} - c'[\theta]]Y. \tag{15}$$

This we can write

$$[\theta]Y = \frac{x}{A}([\theta] - B)Y, \tag{16}$$

where

$$A = (c - \lambda c')/(a - \lambda a'), \qquad B = (b - \lambda b')/(a - \lambda a').$$

Now in (15)

$$\lambda_n = (a[n] - b)/(a'[n] - b'),$$

$$A_n = -((ca' - c'a)[n] + (bc' - b'c))/(ab' - a'b), \qquad B_n = [n].$$

(16) can be written as

$$(xB_n/(x - A_n))f(x) = [\theta]f(x),$$

writing $f(x)$ for Y_n. Or,

$$(A_n - x)f(qx) = [A_n - x\{1 + (q-1)B_n\}]f(x),^2$$

$$f(x) = \sum_{r=-\infty}^{\infty} \frac{[1 - (1/(1+(q-1)B_n))]_r}{[1-q]_r}\left(\frac{1+(q-1)B_n}{A_n}x\right)^r$$

$$(17) \qquad = {}_1\Phi_0\left[\frac{1}{1+(q-1)B_n}; \frac{1+(q-1)B_n}{A_n}x\right]$$

$$= {}_1\Phi_0[q^{-n}; q^n(x/A_n)], \text{ since } B_n = [n]$$

$$= [1 - (x/A_n)]_n \equiv Y_n, \text{ a polynomial of degree } n \text{ in } x,$$

where $[1-\alpha]_n$ denotes the expression $(1-\alpha)(1-q\alpha)\cdots(1-\alpha q^{n-1})$ and

$$ {}_1\Phi_0[\alpha; x] = \sum_{n=0}^{\infty} ([1-\alpha]_n/[1-q]_n)x^n.$$

The type $\mathfrak{A}_1$ is quickest dealt with by writing $[\theta-1]$ for $[\theta]$ in (15). This multiplies the solution Y_n by x and replaces n in (17) by $n-1$, and so

$$Y_n = x[1 - x/A_{n-1}]_{n-1}.$$

References

1. T. W. Chaundy, *Differential equations with polynomial solutions*, Quart. J. Math. Oxford Ser. **20** (1949), 105–120.
2. W. Hahn, *Über die höheren Heineschen Reihen und eine einheitliche Theorie der sogenannten speziellen Funktionen*, Math. Nachr. **3** (1950), 257–294.

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[2] See Hahn [2].