TERM-BY-TERM DIFFERENTIABILITY OF MERCER'S EXPANSION

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Let $K(x, y)$, $0 \leqq x, y \leqq 1$, be a real, symmetric, continuous and non-negative-definite kernel on $[0, 1] \times [0, 1]$. Thus, the integral operator generated by K has nonnegative eigenvalues and the orthonormalized eigenfunctions λ_i and ϕ_i, $i = 0, 1, 2, \cdots$. Then, according to Mercer's theorem [**1**],

$$K(x, y) = \sum_i \lambda_i \phi_i(x) \phi_i(y) \tag{1}$$

uniformly on $[0, 1] \times [0, 1]$. This paper concerns with term-by-term differentiability of the above series while retaining the same sense of convergence. In particular, we obtain a condition, explicitly on K, for such differentiability.

THEOREM. *If* $(\partial^{2n}/(\partial x^n \partial y^n))K(x, y)$ *exists and is continuous on* $[0, 1] \times [0, 1]$, *then* $\phi_i^{(n)}$, *the* nth *derivative of* ϕ_i, *exists and is continuous on* $[0, 1]$ *for each* $i = 0, 1, 2, \cdots$, *and*

$$\frac{\partial^{2n}}{\partial x^n \partial y^n} K(x, y) = \sum_i \lambda_i \phi_i^{(n)}(x) \phi_i^{(n)}(y) \tag{2}$$

uniformly on $[0, 1] \times [0, 1]$. *Conversely, if* $\phi_i^{(n)}$ *exists and is continuous on* $[0, 1]$, *and if the series of* (2) *converges uniformly on* $[0, 1] \times [0, 1]$, *then* $(\partial^{2n}/(\partial x^n \partial y^n))K(x, y)$ *exists, is continuous and is equal to the limit of the series.*

PROOF. The method of induction will be used.

(a) *Proof of the first assertion.* First, since $(\partial^{2n}/(\partial x^n \partial y^n))K(x, y)$ exists and is continuous in (x, y), existence and continuity of $\phi_i^{(n)}$ can be readily established by differentiating n times both sides of

$$\phi_i(x) = \frac{1}{\lambda_i} \int_0^1 K(x, y) \phi_i(y) dy, \qquad i = 0, 1, 2, \cdots. \tag{3}$$

For notational simplicity, define for $k = 1, 2, \cdots, n$,

$$K_k(x, y) = \frac{\partial^{2k}}{\partial x^k \partial y^k} K(x, y),$$

$$R_k^{(j)}(x, y) = K_k(x, y) - \sum_{i=0}^{j} \lambda_i \phi_i^{(k)}(x) \phi_i^{(k)}(y).$$

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The following steps will be taken to establish the assertion for $n=1$.

1°. $R_1^{(j)}(x, x) \geqq 0$, $0 \leqq x \leqq 1$, for every j.

Suppose $R_1^{(j)}(x_0, x_0) < 0$ for some $x_0 \in [0, 1]$. Then it follows from continuity of $R_1^{(j)}$ that there exists a neighborhood $x_0 - \delta < x, y < x_0 + \delta$ where $R_1^{(j)}(x, y) < 0$. Thus, from (1),

$$0 > \int\int_{x_0-\delta}^{x_0+\delta} R_1^{(j)}(x, y)dxdy = \sum_{i=j+1}^{\infty} \lambda_i \int_{x_0-\delta}^{x_0+\delta} \phi_i'(x)dx \int_{x_0-\delta}^{x_0+\delta} \phi_i'(y)dy \geqq 0,$$

a contradiction.

2°. The series of (2) with $n=1$ converges uniformly in x for every fixed y and also in y for every fixed x; thus its limit, denoted by $K_1^*(x, y)$, is continuous in x for every fixed y and also in y for every fixed x.

Note $\sum_i \lambda_i |\phi_i'(x)|^2$ converges since its partial sums form a non-decreasing sequence bounded by $K_1(x, x)$ as seen from 1°. Define

$$M = \max_{0 \leqq x \leqq 1} K_1(x, x),$$

which exists since K_1 is continuous by hypothesis. Then, from Cauchy's inequality,

$$\left| \sum_{i=m}^{n} \lambda_i \phi_i'(x)\phi_i'(y) \right|^2 \leqq \sum_{i=m}^{n} \lambda_i |\phi_i'(x)|^2 \sum_{i=m}^{n} \lambda_i |\phi_i'(y)|^2 \tag{4}$$
$$\leqq M \sum_{i=m}^{n} \lambda_i |\phi_i(y)|^2.$$

Hence, $\sum_i \lambda_i \phi_i'(x)\phi_i'(y)$ converges uniformly in x for every fixed y. Similarly, it converges uniformly in y for every fixed x.

3°. $K_1(x, y) = K_1^*(x, y)$.

Note $K_1 = K_1^*$, a.e. $[dxdy]$, since both K_1 and K_1^* are measurable and, from 2° and (1),

$$\int_0^y \int_0^x [K_1(u, v) - K_1^*(u, v)]dudv$$
$$= \int_0^y \int_0^x K_1(u, v)dudv - \sum_i \lambda_i \int_0^x \phi_i'(u)du \int_0^y \phi_i'(v)dv$$
$$= \int_0^y \int_0^x K_1(u, v)dudv - K(x, y) + K(x, 0) + K(0, y) - K(0, 0)$$
$$= 0$$

for every x and y. Then, from Fubini's theorem [2], for almost every x, $K_1(x, y)=K_1^*(x, y)$ for almost every y. But, since for every fixed x both K_1 and K_1^* are continuous in y, for almost every x the equality holds for every y. Hence, for every y the equality holds for almost every x. However, for every fixed y K_1 and K_1^* are continuous in x also. Thus, the equality holds for every x and y.

4°. The series of (2) with $n=1$ converges uniformly in x and y simultaneously.

From 3°,

$$K_1(x, x) = \sum_i \lambda_i \left| \phi_i'(x) \right|^2.$$

Observe that the partial sums of the series form a nondecreasing sequence of continuous functions converging to a continuous function. Hence, according to Dini's theorem, the convergence is uniform. Then, by applying Cauchy's inequality (4) again, we conclude that $\sum_i \lambda_i\phi_i'(x)\phi_i'(y)$ converges uniformly in x and y simultaneously.

Next, note in the preceding proof for $n=1$ that we have used only the continuity of ϕ_i and uniform convergence of (1) together with $\lambda_i \geqq 0$, $i=0, 1, 2, \cdots$, but not the orthonormality of $\{\phi_i\}$. Hence, upon replacement of ϕ_i, K, ϕ_i', K_1, K_1^* and $R_1^{(j)}$ by $\phi_i^{(k)}$, K_k, $\phi_i^{(k+1)}$, K_{k+1}, K_{k+1}^* and $R_{k+1}^{(j)}$ respectively, the preceding proof establishes the assertion for $n=k+1$ if it holds for $n=k$. Therefore, by induction, the assertion holds for every n.

(b) *Proof of the converse statement.* To prove for $n=1$, note that $K_1^*(x, y)$ is continuous in both x and y since, by hypothesis, the series of (2) with $n=1$ converges uniformly in x and y simultaneously. Note also that

$$\text{(5)} \quad \int_0^y \int_0^x K_1^*((u, v)dudv = \sum_i \lambda_i \int_0^x \phi_i'(u)du \int_0^y \phi_i'(v)dv$$
$$= K(x, y) - K(x, 0) - K(0, y) + K(0, 0),$$

where the second equality follows from (1). Now, from (3), differentiability of ϕ_i implies that of $K(x, 0)$ and $K(0, y)$. Thus, differentiability of the left-hand side of (5) with respect to y and then x, implies existence of $(\partial^2/(\partial x\partial y))K(x, y)$. Hence, upon differentiation of both sides of (5),

$$K_1^*(x, y) = \frac{\partial^2}{\partial x \partial y} K(x, y).$$

Through a similar argument, we establish the converse statement for $n=k+1$ if it holds for $n=k$. Hence, by induction, it holds for every n.

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SOME GENERALIZATIONS OF OPIAL'S INEQUALITY

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The inequality $\int_0^a |uu'| \leqq a/2\int_0^a |u'|^2$ which is valid for absolutely continuous u with $u(0)=0$ has received successively simpler proofs by Opial, [5], Olech [4], Beesack [1], Levinson [2], Pederson [6], and Mallows [3]. It is the purpose of this paper to use the method of Olech to obtain some more general inequalities.

Theorem 1. *Let u be absolutely continuous on (a, b) with $u(a)=0$, where $-\infty \leqq a<b<\infty$. Let $f(t)$ be a continuous, complex function defined for all t in the range of u and for all real t of the form $t(s) = \int_a^s |u'(x)|dx$. Suppose that $|f(t)| \leqq f(|t|)$, for all t, and that $f(t_1) \leqq f(t_2)$ for $0 \leqq t_1 \leqq t_2$. Let r be positive, continuous and in $L^{1-q}[a, b]$, where $1/p+1/q=1$, $p>1$. Let $F(s)=\int_0^s f(x)dx$, $s>0$. Then*

$$\int_a^b |f(u)u'| \, dx \leqq F\left[\left(\int_a^b r^{1-q}\right)^{1/q}\left(\int_a^b r|u'|^p\right)^{1/p}\right]$$

with equality iff $u(x)=A\int_a^x r^{1-q}$. The same result (but with equality for $u(x)=\int_x^b r^{1-q}$) holds if $u(b)=0$ and $-\infty<a<b\leqq\infty$.

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