REMARK ON INVARIANT MEANS

ROBERT KAUFMAN

In this note G is an abelian group and m is generically an invariant mean in G, as defined, for example, in [**4**]. Probabilistic arguments [Baire's theorem] are applied to the measure [topological] space 2^G to obtain information about the means m. One result, which appears to be new, is an answer to a problem set by R. G. Douglas [**2**]: If $2G$ is infinite, not every invariant mean m is inversion invariant.

The space 2^G. The family $\{S\}$ of subsets of G may be identified in the familiar way with the set of all functions on G to $\{0, 1\}$ and provided with the product topology; 2^G is metrizable if G is countable. Since $\{0, 1\}$ is a probability space (the details may safely be suppressed), 2^G may be provided with the product measure μ, even if G is uncountable; it is sufficient in the present case to regard this measure μ as a Baire measure in 2^G. For details of the construction see [**3**, §38]. As to the way this measure is actually used here, we observe that if F is a subset of G containing exactly n elements, $\mu\{S: F\subseteq S\} = 2^{-n}$.

LEMMA 1. *Let f be a bounded real function on G such that for every finite set $F=\{a_1, \cdots, a_n\}$ in G (n may depend on F),* $\sup \sum_1^n f(x+a_i) \geqq n$. *Then for some invariant mean m, $m(f)\geqq 1$.*

PROOF. Let $B(G)$ be the Banach space of real bounded functions on G and $B_0(G)$ the subspace generated by functions h_a-h; by definition $h_a(x)=h(x+a)$. Let N be the set of nonpositive functions in $B(G)$, and for each g in $B(G)$ define $\omega(g)$ to be the norm-distance of g from the convex set $B_0(G)+N$. Then ω is subadditive and positive-homogeneous, while the argument in [**4**, §17.5] shows that $\omega(f)\geqq 1$. By the Hahn-Banach theorem there is a linear functional λ on $B(G)$ for which $\omega\geqq\lambda$ and $\omega(f)\geqq 1$. Then λ is positive, translation invariant and has norm at most 1 as required.

COROLLARY. *If S and T are subsets of G, in order that there exist an invariant mean m such that $m(S)=1$, $m(T)=0$, it is necessary and sufficient that to each finite set $F\subseteq G$ there exist x such that $x+F\subseteq S$, $(x+F)\cap T=\varnothing$.* (Here $m(S)=m(\xi_S)$ for $S\subseteq G$.)

PROOF. For the necessity, let $F=\{a_1, \cdots, a_n\}$ and observe that

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the set $\bigcap_{i=1}^{n} (S-a_i)\sim\bigcup_{1=1}^{n} (T-a_i)$ has m measure 1 and is thus nonempty. For the sufficiency, apply the previous lemma to the function $f=\xi_S-\xi_T$, observing that $m(S)$, $m(T)\in[0, 1]$.

LEMMA 2a. *Let F be a finite subset of G and $U=\{(S, T)\in 2^G\times 2^G$: for some x, $x+F\not\subseteq S$, $(x+F)\cap T=\varnothing\}$. Then U is an open set; if G is infinite, U is dense and contains an open Baire set of $\mu\times\mu$ measure* 1.

PROOF. Since F is finite, U is presented as the union of open sets and is thus open. We now assume G is infinite and choose a sequence $\{x_i: i\geqq 1\}$ in G such that $x_i-x_j\notin F-F$ if $i\neq j$. The complement of U belongs to the closed Baire set

$$\bigcap_{i=1}^{\infty} \{(S, T)\in 2^G\times 2^G: x_i+F\not\subseteq S \quad\text{or}\quad (x_i+F)\cap T\neq\varnothing\}.$$

For each element x of G let $Y_x(S, T)$ be the x-coordinate of S and $Z_x(S, T)$ the x-coordinate of T. The random variables Y_x, Z_x, $x\in G$, are jointly independent [3, §45] for the measure $\mu\times\mu$, and consequently the sets enclosed in braces are jointly independent for distinct indices i. Since each of these sets has the same measure <1, the intersection has measure 0. (If F has r elements, the measure of each set in braces is exactly $1-2^{-2r}$.) Moreover U is dense, since any open subset in $2^G\times 2^G$ contains an open Baire set of positive measure.

COROLLARY. *If G is countably infinite, then for almost all pairs (S, T) there is an invariant mean m such that $m(S)=1$, $m(T)=0$, and an invariant mean m' such that $m'(T)=1$, $m'(S)=0$. The pairs with this property are a dense G_δ.*

PROOF. Observe that the finite subsets of G may be enumerated and apply Lemma 2a and the Corollary to Lemma 1.

LEMMA 2b. *If F is a finite subset of G then $V=\{S\in 2^G: x+F\subseteq S$, $(x+F)\cap -S=\varnothing$ for some x in $G\}$ is open. If $2G$ is infinite, V is dense and contains an open Baire set of μ measure* 1.

PROOF. That V is open is clear. If $2G$ is infinite there is a sequence $\{x_i: i\geqq 1\}$ in G such that $x_i+x_j\notin F-F$ for all $i, j\geqq 1$ and $x_i-x_j\notin F-F$ for $i>j\geqq 1$. The complement of V is contained in the closed Baire set

$$\bigcap_{i=1}^{\infty} \{S\in 2^G: x_i+F\not\subseteq S \quad\text{or}\quad -(x_i+F)\cap S\neq\varnothing\}.$$

The proof now follows that of Lemma 2a.

COROLLARY. *If G is countably infinite and $2G$ is infinite, the sets $S \in 2^G$ for which $m(S)=1$, $m(-S)=0$ for some invariant mean m form a dense G_δ of measure* 1.

THEOREM 3. *If G is infinite, there is more than one invariant mean for G. If $2G$ is infinite, G has invariant means which are not inversion invariant.*

PROOF. If G is infinite, let ϕ be a homomorphism of G onto a countably infinite group. An invariant mean on $\phi(G)$ may be construed as an invariant mean on the field of subsets of G generated by cosets of the kernel of ϕ. According to [**4**, §17.14] any such set function admits an extension to an invariant mean. Since $\phi(G)$ has many invariant means, so also does G. The existence of many invariant means was first proved by Day [**1**].

If $2G$ is infinite, let ψ be a homomorphism of $2G$ onto a countable infinite group, H a countable divisible group containing $\psi(2G)$ and ϕ, an extension of ψ, which maps G into H. Then $\phi(G)$ is countable, $2\phi(G)=\phi(2G)$ is infinite. Because $\phi(G)$ has invariant means which are not inversion invariant, so also does G, by the argument just stated.

To obtain the converse to the second statement of the theorem, set $f^\sim(x)=f(-x)$ and $2G=\{a_1, \cdots, a_n\}$, $n<\infty$. For a bounded function f and invariant mean m

$$m(f^\sim) = m\left(\frac{1}{n}\sum_{i=1}^{n} f^\sim_{a_i}\right);$$

$$\frac{1}{n}\sum_{i=1}^{n} f^\sim_{a_i}(x) = \frac{1}{n}\sum_{i=1}^{n} f(-x-a_i).$$

The identity $-x-2G=x+2G$ shows that

$$\frac{1}{n}\sum_{i=1}^{n} f^\sim_{a_i} = \frac{1}{n}\sum_{i=1}^{n} f_{a_i}$$

and so $m(f^\sim)=m(f)$.

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UNIVERSITY OF ILLINOIS