REMARK ON INVARIANT MEANS
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In this note \( G \) is an abelian group and \( m \) is generically an invariant mean in \( G \), as defined, for example, in [4]. Probabilistic arguments [Baire's theorem] are applied to the measure [topological] space \( 2^G \) to obtain information about the means \( m \). One result, which appears to be new, is an answer to a problem set by R. G. Douglas [2]:

If \( 2G \) is infinite, not every invariant mean \( m \) is inversion invariant.

**The space** \( 2^G \). The family \( \{S\} \) of subsets of \( G \) may be identified in the familiar way with the set of all functions on \( G \) to \( \{0, 1\} \) and provided with the product topology; \( 2^G \) is metrizable if \( G \) is countable. Since \( \{0, 1\} \) is a probability space (the details may safely be suppressed), \( 2^G \) may be provided with the product measure \( \mu \), even if \( G \) is uncountable; it is sufficient in the present case to regard this measure \( \mu \) as a Baire measure in \( 2^G \). For details of the construction see [3, §38]. As to the way this measure is actually used here, we observe that if \( F \) is a subset of \( G \) containing exactly \( n \) elements, \( \mu \{S: F \subseteq S\} = 2^{-n} \).

**Lemma 1.** Let \( f \) be a bounded real function on \( G \) such that for every finite set \( F = \{a_1, \ldots, a_n\} \) in \( G \) (\( n \) may depend on \( F \)), \( \sup \sum_{i=1}^{n} f(x+a_i) \geq n \). Then for some invariant mean \( m \), \( m(f) \geq 1 \).

**Proof.** Let \( B(G) \) be the Banach space of real bounded functions on \( G \) and \( B_0(G) \) the subspace generated by functions \( h_a - h \); by definition \( h_a(x) = h(x+a) \). Let \( N \) be the set of nonpositive functions in \( B(G) \), and for each \( g \) in \( B(G) \) define \( \omega(g) \) to be the norm-distance of \( g \) from the convex set \( B_0(G) + N \). Then \( \omega \) is subadditive and positive-homogeneuous, while the argument in [4, §17.5] shows that \( \omega(f) \geq 1 \). By the Hahn-Banach theorem there is a linear functional \( \lambda \) on \( B(G) \) for which \( \omega(F) \geq \lambda(f) \). Then \( \lambda \) is positive, translation invariant and has norm at most 1 as required.

**Corollary.** If \( S \) and \( T \) are subsets of \( G \), in order that there exist an invariant mean \( m \) such that \( m(S) = 1 \), \( m(T) = 0 \), it is necessary and sufficient that to each finite set \( F \subseteq G \) there exist \( x \) such that \( x + F \subseteq S \), \( (x + F) \cap T = \emptyset \). (Here \( m(S) = m(\xi_S) \) for \( S \subseteq G \).

**Proof.** For the necessity, let \( F = \{a_1, \ldots, a_n\} \) and observe that

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the set \( \bigcap_{i=1}^{n} (S - a_i) \sim \bigcup_{i=1}^{n} (T - a_i) \) has \( m \) measure 1 and is thus non-empty. For the sufficiency, apply the previous lemma to the function \( f = \xi_S - \xi_T \), observing that \( m(S), m(T) \in [0, 1] \).

**Lemma 2a.** Let \( F \) be a finite subset of \( G \) and \( U = \{(S, T) \in 2^G \times 2^G : \text{for some } x, x + F \subseteq S, (x + F) \cap T = \emptyset \} \). Then \( U \) is an open set; if \( G \) is infinite, \( U \) is dense and contains an open Baire set of \( \mu \times \mu \) measure 1.

**Proof.** Since \( F \) is finite, \( U \) is presented as the union of open sets and is thus open. We now assume \( G \) is infinite and choose a sequence \( \{x_i : i \geq 1\} \) in \( G \) such that \( x_i - x_j \notin F - F \) if \( i \neq j \). The complement of \( U \) belongs to the closed Baire set

\[
\bigcap_{i=1}^{\infty} \left\{ (S, T) \in 2^G \times 2^G : x_i + F \not\subseteq S \text{ or } (x_i + F) \cap T \neq \emptyset \right\}.
\]

For each element \( x \) of \( G \) let \( Y_x(S, T) \) be the \( x \)-coordinate of \( S \) and \( Z_x(S, T) \) the \( x \)-coordinate of \( T \). The random variables \( Y_x, Z_x, x \in G \), are jointly independent [3, §45] for the measure \( \mu \times \mu \), and consequently the sets enclosed in braces are jointly independent for distinct indices \( i \). Since each of these sets has the same measure \(<1\), the intersection has measure 0. (If \( F \) has \( r \) elements, the measure of each set in braces is exactly \( 1 - 2^{-2r} \).) Moreover \( U \) is dense, since any open subset in \( 2^G \times 2^G \) contains an open Baire set of positive measure.

**Corollary.** If \( G \) is countably infinite, then for almost all pairs \((S, T)\) there is an invariant mean \( m \) such that \( m(S) = 1 \), \( m(T) = 0 \), and an invariant mean \( m' \) such that \( m'(T) = 1 \), \( m'(S) = 0 \). The pairs with this property are a dense \( G_\delta \).

**Proof.** Observe that the finite subsets of \( G \) may be enumerated and apply Lemma 2a and the Corollary to Lemma 1.

**Lemma 2b.** If \( F \) is a finite subset of \( G \) then \( V = \{S \in 2^G : x + F \subseteq S, (x + F) \cap -S = \emptyset \text{ for some } x \text{ in } G \} \) is open. If \( 2G \) is infinite, \( V \) is dense and contains an open Baire set of \( \mu \) measure 1.

**Proof.** That \( V \) is open is clear. If \( 2G \) is infinite there is a sequence \( \{x_i : i \geq 1\} \) in \( G \) such that \( x_i + x_j \notin F - F \) for all \( i, j \geq 1 \) and \( x_i - x_j \notin F - F \) for \( i > j \geq 1 \). The complement of \( V \) is contained in the closed Baire set

\[
\bigcap_{i=1}^{\infty} \left\{ S \in 2^G : x_i + F \not\subseteq S \text{ or } -(x_i + F) \cap S \neq \emptyset \right\}.
\]

The proof now follows that of Lemma 2a.
Corollary. If $G$ is countably infinite and $2G$ is infinite, the sets $S \subseteq 2^G$ for which $m(S) = 1$, $m(-S) = 0$ for some invariant mean $m$ form a dense $G_6$ of measure 1.

Theorem 3. If $G$ is infinite, there is more than one invariant mean for $G$. If $2G$ is infinite, $G$ has invariant means which are not inversion invariant.

Proof. If $G$ is infinite, let $\phi$ be a homomorphism of $G$ onto a countably infinite group. An invariant mean on $\phi(G)$ may be construed as an invariant mean on the field of subsets of $G$ generated by cosets of the kernel of $\phi$. According to [4, §17.14] any such set function admits an extension to an invariant mean. Since $\phi(G)$ has many invariant means, so also does $G$. The existence of many invariant means was first proved by Day [1].

If $2G$ is infinite, let $\psi$ be a homomorphism of $2G$ onto a countable infinite group, $H$ a countable divisible group containing $\psi(2G)$ and $\phi$, an extension of $\psi$, which maps $G$ into $H$. Then $\phi(G)$ is countable, $2\phi(G) = \phi(2G)$ is infinite. Because $\phi(G)$ has invariant means which are not inversion invariant, so also does $G$, by the argument just stated.

To obtain the converse to the second statement of the theorem, set $f^\sim(x) = f(-x)$ and $2G = \{a_1, \ldots, a_n\}, n < \infty$. For a bounded function $f$ and invariant mean $m$

$$m(f^\sim) = m\left(\frac{1}{n} \sum_{i=1}^{n} f_{a_i}^\sim\right);$$

$$\frac{1}{n} \sum_{i=1}^{n} f_{a_i}^\sim(x) = \frac{1}{n} \sum_{i=1}^{n} f(-x - a_i).$$

The identity $-x - 2G = x + 2G$ shows that

$$\frac{1}{n} \sum_{i=1}^{n} f_{a_i}^\sim = \frac{1}{n} \sum_{i=1}^{n} f_{a_i},$$

and so $m(f^\sim) = m(f)$.

References


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