A NOTE ON LIMIT THEOREMS FOR MARKOV BRANCHING PROCESSES[1]

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Introduction. Let $\{Z'_n, n \geqq 0\}$, $Z_0 = 1$, denote the random variables in a Galton-Watson process with generator

$$f(s) = E[s^{Z_1}] = \sum_{k=0}^{\infty} p_k s^k, \quad |s| \leqq 1, \tag{1}$$

where p_k is the probability for a particle in the nth generation to produce k particles in the $(n+1)$th generation, independently of n.

Let $\{Z'_t, t \geqq 0\}$, $Z'_0 = 1$, denote the random variables in a continuous-parameter Markov branching process with generator

$$\mathcal{E}(s) = b\left[\sum_{k=0}^{\infty} q_k s^k - s\right], \quad |s| \leqq 1, \tag{2}$$

where $b\Delta + O(\Delta)$ is the probability for a particle existing at time t to die in the interval $(t, t+\Delta)$, independently of t, and each $q_k (q_1 \equiv 0)$ is the probability for a particle dying at time τ to produce k particles at time τ, independently of τ.

These assumptions imply that $\{Z_n, n \geqq 0\}$ and $\{Z'_t, t \geqq 0\}$ are time homogeneous Markov branching processes.

The purpose of this note is to show how the fundamental limit theorems for continuous-parameter Markov branching processes can be derived using the corresponding theorems for Galton-Watson processes. In a brief summary, each limit theorem is a statement about the limiting properties of certain functions, of the Z_n or Z'_t-processes, whose values are probabilities, generating functions or distribution functions. In a natural way, these values associate to each limit theorem a metric space in which the limiting behavior is studied. Using the metric space as a common reference, it is possible to relate the limiting behavior of a Z'_t-process to that of a Z_n-process.

This relation is achieved by the following result (first proved by J. F. C. Kingman in the special case $I = (0, \infty)$, [1, p. 594–597]):

(K) *Let π be a continuous mapping of $[0, \infty)$ into a metric space X.*

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Assume that for each t in $(0, \infty)$ π has either a right- or left-limit at t. Let $I = \{a < t < b\}$, $0 \leqq a < b \leqq \infty$. Assume that for each $\Delta \in I$ the sequence $\pi(n\Delta)$ converges to $x(\Delta) \in X$ as $n \to \infty$. Then

(i) *$x(\Delta)$ does not depend on $\Delta \in I$; i.e., $x(\Delta) = x_0$ for all $\Delta \in I$,*

(ii) *$\pi(t)$ converges to x_0 as $t \to \infty$.*

PROOF. The essential fact in this proof is the following property of real numbers, H. T. Croft [2] and J. F. C. Kingman [1].

(C) *Assume U_k, $k = 1, 2, 3, \cdots$, are open unbounded subsets and I is an open interval, all in $[0, \infty)$. Then there exists a real number $\Delta_0 \in I$ such that, for each U_k, $n\Delta_0 \in U_k$ for infinitely many positive integers n.*

Assume $\Delta_1, \Delta_2 \in I$ are such that $x(\Delta_1) = \lim_n \pi(n\Delta_1) \neq \lim_n \pi(n\Delta_2) = x(\Delta_2)$. Then, the convergence of $\pi(n\Delta)$ and the right- or left-limit-properties of π imply the existence of two disjoint neighborhoods N_1 and N_2, of $x(\Delta_1)$ and $x(\Delta_2)$ respectively, and two unbounded open subsets $U_1, U_2 \subseteq [0, \infty)$ such that, for $t \in U_j$, $\pi(t) \in N_i$, $i = 1, 2$. Applying property (C) to U_1, U_2 and I, there exists a $\Delta_0 \in I$ such that $n\Delta_0$ is in each of U_1 and U_2 for infinitely many values of n. This contradicts the convergence of $\pi(n\Delta_0)$ to $x(\Delta_0)$. Therefore for some $x_0 \in X$ and all $\Delta \in I$, $\lim_{n\to\infty} \pi(n\Delta) = x_0$.

For the same reasons, the nonconvergence of $\pi(t)$ to x_0, as $t \to \infty$, would imply the existence of an open unbounded set $U \subseteq [0, \infty)$ and a neighborhood N of x_0 such that $\pi(t) \notin N$ for all $t \in U$. Applying (C) to U and I gives the existence of a $\Delta_0 \in I$ such that $n\Delta_0 \in U$ for infinitely many values of n. However this would contradict the convergence of $\pi(n\Delta_0)$ to x_0; so that, $\lim_{t\to\infty} \pi(t) = x_0$, proving (K).

In the place of an exhaustive development of this method, we have decided to illustrate its applicability by extending some of the fundamental results for Galton-Watson processes.

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Preliminaries. Again we let $f(s)$, (1), and $\mathcal{E}(s)$, (2), be the respective generators for the Z_n and Z_t'-processes. We list for convenient reference the following moment calculations:

$$E[Z_n] = \rho^n \tag{3}$$

where $\rho = f'(1)$.

$$E[Z_t'] = e^{at} \tag{4}$$

where $a = \mathcal{E}'(1)$.

$$\operatorname{Var}[Z_n] = \begin{cases} \operatorname{Var}[Z_1] \dfrac{\rho^n(\rho^n - 1)}{\rho^2 - \rho}, & \rho \neq 1, \\ n \operatorname{Var}[Z_1], & \rho = 1, \end{cases} \tag{5}$$

$$\operatorname{Var}[Z_t'] = \begin{cases} \dfrac{\mathcal{E}''(1) - \mathcal{E}'(1)}{\mathcal{E}'(1)} e^{at}(e^{at} - 1), & a \neq 0, \\ \mathcal{E}''(1)t, & a = 0. \end{cases} \tag{6}$$

We also have need for some knowledge about the Z_t-process which allows us to translate weighted moment conditions on its generator into the same conditions on Z_t. The following is adequate for the purpose of this note.

PROPOSITION 1. *Let $\{q_k\}_{k \neq 1}$ be the infinitesimal branching probabilities,* (2), *for the Z_t'-process. Then for arbitrary positive integer α,*

$$\sum_{k=2}^{\infty} (k^\alpha \log k) q_k < \infty \Leftrightarrow \sum_{k=2}^{\infty} (k^\alpha \log k) P[Z_\Delta' = k] < \infty, \quad \Delta > 0.$$

(For a coherence in presentation, we have chosen to postpone giving a proof of this proposition until the Appendix of this note.)

Limit theorems. A general type of limiting behavior for a Galton-Watson process with $\rho > 0$, (3), is described by the normalized process

$$W_n = \rho^{-n} Z_n. \tag{7}$$

Since $\{W_n, n \geqq 0\}$ is a nonnegative martingale with $E[W_n] = 1$ [**3**, p. 14], we have always the existence with probability 1 of

$$\lim_{n \to \infty} W_n = W. \tag{8}$$

However when $\rho \leqq 1$, W is identically 0. Unless further restrictions are placed on $f(s)$, this degeneracy can also occur when $\rho > 1$.

A complete description of the case $\rho > 1$ is given by the following:

THEOREM 1 (B. STIGUM AND H. KESTEN, 1965, TO BE PUBLISHED, [**4**]). *Assume $\rho > 1$. Let W_n and W be defined by* (7) *and* (8).

$$(9)\quad \begin{aligned} &\text{(a) } \textit{If } \sum_{k\geqq 2} (k\log k)p_k = \infty, \qquad \textit{then } E[W] = 0.\\ &\text{(b) } \textit{If } \sum_{k\geqq 2} (k\log k)p_k < \infty, \qquad \textit{then } E[W] = 1,\end{aligned}$$

and the probability distribution of W

(i) *either has a continuous density except possibly for a discontinuity at the origin,*

(ii) *or it is concentrated at one point.*

In the continuous parameter process with a as defined in (4),

$$(10)\qquad W_t' = e^{-at}Z_t', \qquad a = b[h'(1) - 1],$$

is also a nonnegative martingale with $E[W_t']=1$ [**3**, p. 108], and

$$(11)\qquad \lim_{t\to\infty} W_t' = W'$$

exists with probability 1. Again there is the possibility of W' being degenerate.

Our description of the continuous parameter case with $a>0$ is

THEOREM 1′. *Assume $a>0$. Let W_t' and W' be defined by* (10) *and* (11).

(a) *If $\sum_{k\geqq 2}(k\log k)q_k=\infty$, then $E[W']=0$.*

(b) *If $\sum_{k\geqq 2}(k\log k)q_k<\infty$, then $E[W']=1$*

and the probability distribution of W'

(i) *either has a continuous density except possibly for a discontinuity at the origin,*

(ii) *or it is concentrated at one point.*

PROOF. It is sufficient to restrict our attention to the family of distribution functions defined by

$$(12)\qquad H(t,u) = P[e^{-at}Z_t' \leqq u] = \sum_{k\leqq \exp(at)u} P[Z_t' = k] \quad (t \geqq 0).$$

Let us examine $H(t,u)$ more carefully. Assuming u is a point of continuity for $H(\tau,u)$, there exists a $\delta>0$ such that the interval $(e^{at}u-\delta, e^{a\tau}u+\delta)$ contains no integer since the Z_t'-process is aperiodic. Now consider when $t>\tau$ ($t<\tau$ is similarily handled). Then

$$(13)\quad \begin{aligned} H(t,u) - H(\tau,u) \leqq &\sum_{k\leqq \exp(a\tau)u} \left| P[Z_t' = k] - P[Z_\tau' = k]\right| \\ &+ \sum_{\exp(a\tau)u\leqq k\leqq \exp(at)u} P[Z_t' = k].\end{aligned}$$

The first summation in (13) involves a finite number of terms, and therefore, using the continuity of each $P[Z_t'=k]$ in t, tends to 0 as t

tends to τ. The second summation in (13) vanishes when $|e^{at}u - e^{a\tau}u| < \delta$. Therefore for each point u of continuity of $H(t, u)$, the left-hand side of (13) tends to 0 as t tends to τ.

This type of convergence for distribution functions is equivalent to convergence in the metric space $\mathfrak{L}[0 \leqq u < \infty]$ of distribution functions H on $[0, \infty)$ with the Levy metric

$$d(H_1, H_2) = \inf\{\epsilon \mid H_1(u - \epsilon) - \epsilon \leqq H_2(u) \leqq H_1(u + \epsilon) + \epsilon\} \text{ [5, p. 33].}$$

Therefore the mapping π of $(0, \infty)$ into $\mathfrak{L}[0 \leqq u < \infty]$, defined by

$$\pi(t) = H(t, u),$$

is continuous. Consequently, we can use (K) with $X = \mathfrak{L}[0 \leqq u < \infty]$, π as defined above and $I = (0, \infty)$ to reduce the problem to a study of the convergence in $\mathfrak{L}[0 \leqq u < \infty]$ as $n \to \infty$ of the distribution functions, $H(n\Delta, u)$, associated with the $Z'_{n\Delta}$-process.

For each Δ in I, $\{Z'_{n\Delta}; n \geqq 0\}$ is a Galton-Watson process with generator

$$f^{(\Delta)}(s) = \sum_{k=0}^{\infty} P[Z'_\Delta = k]s^k, \qquad |s| \leqq 1.$$

The moment calculations (3) and (4) show $\rho^{(\Delta)} \equiv E[Z'_\Delta] = e^{a\Delta}$; and so, the assumption $a > 0$ implies $\rho^{(\Delta)} > 1$. Moreover, the distribution functions for the normalized process $(\rho^{(\Delta)})^{-n} Z'_{n\Delta}$, (7), are precisely $H(n\Delta, u)$. Applying Proposition 1, we have $\sum_{k=2}^{\infty}(k \log k)q_k < \infty \Leftrightarrow E[Z'_\Delta \log Z_\Delta'] < \infty$, $\Delta > 0$.

Therefore, each $Z'_{n\Delta}$-process satisfies the conditions of Theorem 1. Using Theorem 1 we can assert that, for each Δ in I, $H(n\Delta, u)$ converges as $n \to \infty$ to a nondegenerate distribution $H^{(\Delta)}(u)$ satisfying either (9) (a) or (b).

Applying (K), we know $H^{(\Delta)}(u)$ does not depend on Δ and setting $H(u) = H^{(\Delta)}(u)$, we can conclude that $H(t, u)$ converges in the metric of $\mathfrak{L}[0 \leqq u < \infty]$ to $H(u)$ as $t \to \infty$, completing the proof.

Our next result is for Z_n-processes with $\rho < 1$.

THEOREM 2 (A. JOFFE, 1965, [6]). *If $f'(1) < 1$, the conditional generating functions*

$$g_n(s) = \frac{f_{n(s)} - f_{n(0)}}{1 - f_n(s)}, \qquad f_n(s) = E[s^{Z_n}],$$

converge uniformly in any closed region, interior to $\{|s| = 1\}$, to a limit $g(s)$ which is analytic for $|s| < 1$, continuous at $s = 1$ with $g(1) = 1$ and is therefore a generating function. In particular

$$\lim_{n\to\infty} P[Z_n = k \mid Z_n > 0] = \textit{coefficient of } s^k \textit{ in } g(s).$$

Also, $E[Z_1 \log Z_1] < \infty$ *is a necessary and sufficient condition for* $g'(1) < \infty$.

(A. Joffe has in his paper a different necessary and sufficient condition for $g'(1) < \infty$; however, F. Spitzer has shown the two conditions to be equivalent.)

The convergence properties stated in Theorem 2 show the convergence of g_n to g with respect to $\max_{0\leq\theta\leq 2\pi} |g_n(e^{i\theta}) - g(e^{i\theta})|$. Our corresponding theorem for the continuous parameter process is

THEOREM 2′. *If* $a = \mathcal{E}'(1) < 0$, *the conditional generating functions*

$$g(t, s) = \frac{F(t, s) - F(t, 0)}{1 - F(t, 0)}, \qquad 0 \leqq t < \infty, \ |s| \leqq 1,$$

where $F(t, s) = E[s^{Z'_t}]$, *converge uniformly in any closed region, interior to* $\{|s| = 1\}$, *to a generating function* $g(s)$. *In particular*

$$\lim_{n\to\infty} P[Z'_t = k \mid Z'_t > 0] = \textit{coefficient of } s^k \textit{ in } g(s).$$

Moreover, $\sum_{k=2}^{\infty} (k \log k) q_k < \infty$ *is a sufficient condition for* $g'(1) < \infty$.

PROOF. The generating functions

$$F(t, s) = \sum_{k=0}^{\infty} P[Z'_t = k] s^k, \qquad |s| \leqq 1, \ t > 0,$$

can be characterized as the unique solution to

$$F(t, s) = se^{-bt} + \int_0^t h[F(t - \tau, s)] be^{-b\tau} d\tau \quad \text{and} \quad F(t, 1) = 1,$$
$$0 \leqq t < \infty, \ |s| \leqq 1,$$

where $h(s) = \mathcal{E}(s) - s$, which for each t is a generating function in s. This characterization is sufficient to show $F(t, s)$ is continuous in (t, s) for $0 \leqq t$ and $|s| \leqq 1$. Letting $G[|s| \leqq 1]$ denote the metric space of all generating functions on $|s| \leqq 1$ with the metric $d(g_1, g_2) = \max_{0\leq\theta\leq 2\pi} |g_1(e^{i\theta}) - g_2(e^{i\theta})|$, the mapping π of $[0, \infty)$ into G, defined by $\pi(t) = F(t, s)$, is therefore continuous.

For each $\Delta > 0$, the random variables $\{Z'_{n\Delta}, n \geqq 0\}$ form a Galton-Watson process with generator $f^{(\Delta)}(s) = F(\Delta, s)$. So we again use (K) with $X = G[|s| \leqq 1]$, $\pi(t) = g(t, s)$ and $I = (0, \infty)$ to reduce the problem to the study of the $Z'_{n\Delta}$-process, Δ in I.

The moment calculations (3) and (4) and the assumption $a<0$ imply that $f^{(\Delta)\prime}(1)=e^{a\Delta}<1$. Therefore Theorem 2 can be applied to show the associated conditional generating functions $\pi(n\Delta)=g_n^{(\Delta)}$

$$\left(g_n^{(\Delta)}(s) = \frac{f_n^{(\Delta)}(s) - f_n^{(\Delta)}(0)}{1 - f_n^{(\Delta)}(0)}, \; f_n^{(\Delta)}(s) = E\left[s^{Z'_{n\Delta}}\right]\right)$$

converge in $G[|s| \leqq 1]$ to a generating function $g^{(\Delta)}$. Furthermore, $g^{(\Delta)\prime}(1) < \infty$ if and only if $\sum_{k=2}^{\infty} (k \log k)q_k < \infty$. This follows from Proposition 1 and the second part of Theorem 2.

Applying (K) with X, π and I as defined above, we can conclude that $g^{(\Delta)}$ does not depend on Δ, Δ in I. Therefore setting $g(s)=g^{(\Delta)}(s)$, $|s| \leqq 1$, we can assert that $g(t, s)$ converges in the metric of $G[|s| \leqq 1]$ to $g(s)$ as $t\to\infty$. This establishes Theorem 2′.

Another type of limiting behavior for Galton-Watson processes, with $\rho=1$, is given by

THEOREM 3 (H. KESTEN, P. NEY, F. SPITZER [7, COROLLARY 1, p. 7]). *If $f'(1)=1$ and $f''(1)<\infty$, the conditional distribution functions*

$$H_n(u) \equiv P\left[\frac{2Z_n}{nf''(1)} \leqq u \mid Z_n > 0\right], \qquad 0 \leqq u < \infty,$$

converge for each value of u to $1-e^{-u}$.

The convergence properties stated in Theorem 3 give the weak convergence of the distributions H_n to the exponential distribution e; i.e., $\lim_{n\to\infty} H_n(u)=e(u)$ at each point u of continuity of e, where $e(u)=0$ for $u<0$ and $e(u)=1-e^{-u}$ for $u\geqq 0$. Our extension of Theorem 3 is

THEOREM 3′. *If $a=\mathcal{E}'(1)=1$ and $\mathcal{E}''(1)<\infty$ the conditional distribution functions*

$$H(t, u) \equiv P\left[\frac{2Z_t'}{\mathcal{E}''(1)t} \leqq u \mid Z_t' > 0\right]$$

converge weakly to the exponential distribution $e(u)$.

PROOF. Consider the distribution functions

$$H(t, u) = P\left[\frac{2Z_t'}{\mathcal{E}''(1)t} \leqq u \mid Z_t' > 0\right] = \sum_{k \leqq \frac{1}{2}\mathcal{E}''(1)tu} P[Z_t' = k \mid Z_t' > 0].$$

Reasoning similar to that used in the proof of Theorem 1′ shows the mapping π of $(0, \infty)$ into $\mathcal{L}[0\leqq u<\infty]$, $\pi(t)=H(t, u)$, is continuous in t.

The moment calculations (3)–(6) and the assumptions $a=0$ and $\mathcal{E}''(1)<\infty$ imply that $E[Z'_\Delta]=1$ and $\operatorname{Var}[Z'_\Delta]=\mathcal{E}''(1)\Delta<\infty$; so that for each $\Delta>0$ the Galton-Watson process $\{Z'_{n\Delta},\ n\geqq 0\}$ satisfies the conditions of Theorem 3. In particular

$$H(n\Delta, u) = P\left[\frac{2Z'_{n\Delta}}{nf^{(\Delta)''}(1)} \leqq u \,\Big|\, Z'_{n\Delta} > 0\right] \tag{14}$$

where $f^{(\Delta)}(s)$ is the generator of the $Z'_{n\Delta}$-process, $\Delta>0$.

Therefore each sequence of conditional distribution functions $\{H(n\Delta, u),\ n\geqq 0\}$, $\Delta>0$, converges in the Levy metric to the exponential distribution $e(u)$. Applying (K) with $X=\mathcal{L}[0\leqq u<\infty]$, $\pi(t)=H(t, u)$ and $I(0, \infty)$, we can assert that $\pi(t)$ and $H(t, u)$ converges as $t\to\infty$ in the same sense to $e(u)$, completing the proof for Theorem 3′.

A finer analysis for the case $\rho=1$ is the following:

THEOREM 4 (H. KESTEN, P. NEY, F. SPITZER [7, THEOREM 6, p. 33]). *Assume* $f'(1)=1$ *and* $\sum_{k=2}^\infty (k^2\log k)p_k<\infty$. *Set* d $=\text{g.c.d.}\{k \mid k\geqq 1,\ p_k>0\}$. *Let* $k(n)$ *be an integral-valued function of* n, $n>0$, *with* $k(n)/n$ *bounded. Then*

$$\lim_{n\to\infty}\left(\frac{nf''(1)}{2}\right)^2 \exp\left[\frac{2k(n)}{nf''(1)}\right] P[Z_n = k(n)] = d. \tag{15}$$

Our extension of this result is

THEOREM 4′. *Assume* $a=0$ *and* $\sum_{k=2}^\infty (k^2\log k)q_k<\infty$. *Let* $k(t)$ *be an integral-valued function of* t, $t>0$, *with* $k(t)/t$ *bounded as* $t\to\infty$. *Then*

$$\lim_{t\to\infty}\left(\frac{t\mathcal{E}''(1)}{2}\right)^2 \exp\left[\frac{2k(t)}{t\mathcal{E}''(1)}\right] P[Z'_t = k(t)] = 1. \tag{16}$$

(This result was first given by Čistyakov [8] when $\mathcal{E}''(1)<\infty$.)

PROOF. We define the mapping π of $(0, \infty)$ into the metric space $X=[0, \infty)$ by

$$\pi(t) = \left(\frac{t\mathcal{E}''(1)}{2}\right)^2 \exp\left[\frac{2k(t)}{t\mathcal{E}''(1)}\right] P[Z'_t = k(t)].$$

Our assumption about $k(t)$ and the continuity properties of $P[Z'_t=k(t)]$ imply that $\pi(t)$ has either a right- or left-limit at each $t>0$.

When we consider the family of Galton-Watson processes $\{Z'_{n\Delta};\ n>0\}$ with generators $f^{(\Delta)}(s)=\sum_{k=0}^{\infty}P(Z'_{\Delta}=k)s^k$, our moment assumptions and the result of Proposition 1 imply that each of these processes satisfies the conditions of Theorem 4. Also each $Z'_{n\Delta}$-process is aperiodic. (The Z'_t-process is aperiodic.) Since $\pi(n\Delta)$ equals the argument of the left-hand side of (15), we can then apply Theorem 4 (with $d=1$) and assert the convergence of $\pi(n\Delta)$ as $n\to\infty$ to 1 for each $\Delta>0$. An application of (K) with $X=[0,\ \infty)$, π as defined above and $I=(0,\ \infty)$ is sufficient to determine the convergence asserted in (16).

Appendix. (We shall give only an outline for a proof of Proposition 1.)

Outline of Proof. We first define a sequence $\{Y_t^{(n)},\ t>0\}$ $(n=1,\ 2,\ \cdots)$ of auxiliary processes by

$Y_t^{(n)}=\infty$ if at least one nth generation particle is born at or before time t,
$=Z'_t$ otherwise.

Setting

$$p_k^{(n)}(t)=P[Y_t^{(n)}=k\mid Z'_0=1],$$
$$p_{jk}^{(n)}(t)=P[Y_t^{(n)}=k\mid Z'_0=j]$$

it is easily verified that

$$p_k^{(n+1)}(t)=\sum_{j\geq 2}q_j\int_0^t p_{jk}^{(n)}(t-u)be^{-bu}du,\qquad (k\geqq 2,\ t>0), \tag{17}$$

and

$$p_k^{(n)}(t)\leqq p_k^{(n+1)}(t)\to p_k(t)\qquad \text{as } n\to\infty, \tag{18}$$

uniformly on bounded t-intervals.

Using (17), we have

$$\sum_{k\geq 2}(k^{\alpha}\log k)p_k^{(n+1)}(t)=\sum_{j\geq 2}q_j\int_0^t\sum_{k\geq 2}(k^{\alpha}\log k)p_{jk}^{(n)}(t-u)be^{-bu}du. \tag{19}$$

It is sufficient to show

$$M^{(n)}(t)=\sum_{k\geq 2}(k^{\alpha}\log k)p_k^{(n)}(t)$$

is uniformly bounded for all n on bounded t-intervals.

Using the convexity of $x^\alpha \log x$ and the fact that $Y_t^{(n)}$ given $Z_0'=j$ is distributed as the sum of j independent and identical copies, $Y_t^{(n)}(1), \cdots, Y_t^{(n)}(j)$, of $Y_t^{(n)}$ given $Z_0'=1$, we can write

$$\begin{aligned} E[(Y_t^{(n)})^\alpha \log Y_t^{(n)} \mid Z_0' = j] & \\ &\leqq \frac{1}{j}\sum_{i=1}^{j} E[(jY_t^{(n)}(i))^\alpha \log jY_t^{(n)}(i) \mid Z_0' = 1] \\ &\leqq \sum_{k\geqq 2} (jk)^\alpha \log jk p_k^{(n)}(t) \\ &\leqq (j^\alpha \log j)\sum_{k\geqq 2} k^\alpha p_k^{(n)}(t) + j^\alpha \sum_{k\geqq 2} (k^\alpha \log k) p_k^{(n)}(t). \end{aligned} \tag{20}$$

Substituting (20) into (19) shows

$$\begin{aligned} M^{(n+1)}(t) \leqq \left[\sum_{j\geqq 2} (j^\alpha \log j) q_j\right] \int_0^t \left[\sum_{k\geqq 2} k^\alpha p_k^{(n)}(t-u)\right] be^{-bu} du \\ + \left(\sum_{j\geqq 2} j^\alpha q_j\right) \int_0^t \left[\sum_{k\geqq 2} (k^\alpha \log k) p_k^{(n)}(t-u)\right] be^{-bu} du. \end{aligned}$$

Consequently the finiteness of $M^{(n)}(t)$ for all n follows readily by induction on n from the finiteness of $\sum_{k\geqq 2} (k^\alpha \log k) q_k$.

The same techniques can be used to show $M^{(n)}(t)$ is bounded, uniformly in n, on any bounded t-interval.

The argument is then completed by using the monotone convergence of $p_k^{(n)}(t)$ to $p_k(t)$, stated in (18), to show

$$\sum_{k\geqq 2} (k^\alpha \log k) p_k(t) = \lim_{n\to\infty} M^{(n)}(t) < \infty$$

when $\sum_{k\geqq 2} (k^\alpha \log k) q_k < \infty$. To prove the converse, set $\phi(k) = k^\alpha \log k$ and let τ be the ramification time for the initial particle. Then

$$E\left[\phi\left(\sup_{t\leqq\Delta} Z_t'\right)\right] \geqq E[\phi(Z_\tau'); \tau \leqq \Delta]$$

since $\phi(k)$ is increasing.

Since $P(Z_\tau' = k; \tau \leqq \Delta) = q_k(1-\exp(-b\Delta))$, we have

$$E\left[\phi\left(\sup_{t\leqq\Delta} Z_t'\right)\right] \geqq \sum (k \log k) q_k (1 - \exp(-b\Delta)). \tag{21}$$

It is not difficult to show that

$$E[\phi(Z_\Delta')] < \infty \Rightarrow E\left[\phi\left(\sup_{t\leqq\Delta} Z_t'\right)\right] \leqq \infty. \tag{22}$$

The result follows from (21) and (22).

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