DIFFERENTIABILITY OF SAMPLE FUNCTIONS IN GAUSSIAN PROCESSES[1]

J. YEH

1. Let $\{\xi(t), t\in T\}$, $T=[0, 1]$, be a Gaussian process, defined on an underlying probability space $(X, \mathfrak{F}, P)$, in which X is the collection of all real valued functions $x(t)$ defined on T, $\mathfrak{F}$ is the σ-field of subsets of X generated by the Borel cylinders in X, and $\xi(t)$ is defined by

$$\xi(t, x) = x(t) \quad \text{for } t \in T, x \in X.$$

Here, by a Borel cylinder in X, we mean a subset B_X of X defined by

$$B_X = \{x \in X; [x(t_1), \cdots, x(t_n)] \in B\}$$

where $t_1, \cdots, t_n \in T$, and B is a Borel set in the n-dimensional Euclidean space. Such a process exists according to the Kolmogoroff Extension Theorem. If, further,

$$E\{\xi(t)\} = 0 \quad \text{for } t \in T, \tag{1}$$

and for some $\beta, b>0$,

$$E\{|\xi(t') - \xi(t'')|^2\} \leqq b|t' - t''|^\beta \qquad \text{for } t', t'' \in T \tag{2}$$

and the process is separable then[2]

$$P^*(C_\lambda) = 1 \quad \text{for } 0 < \lambda < \beta/2$$

where P^* is the outer measure of P and C_λ is the subset of X consisting of the Lipschitz λ-continuous elements, i.e., those functions which satisfy

$$|x(t') - x(t'')| \leqq h|t' - t''|^\lambda \quad \text{for } t', t'' \in T$$

with h depending on x.

In this article we consider the differentiability of the sample paths $x(t)$ of the process $\{\xi(t), t\in T\}$. We shall assume that our process satisfies the additional condition that for some $\alpha, a>0$,

$$E\{|\xi(t') - \xi(t'')|^2\} \geqq a|t' - t''|^\alpha \quad \text{for } t', t'' \in T. \tag{3}$$

Our result is stated in the theorem in §2, after a brief discussion of

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[2] A proof for this statement can be found for instance in [4].

the measure induced on the space of continuous functions by the process. Our theorem implies in particular that, if $\alpha/2<1$, then almost every sample function $x(t)$ is almost nowhere differentiable with respect to t on T. This is an extension of the classical result of Wiener for the Brownian motion process.

2. Let C be the subset of X consisting of the continuous functions. Now that $C_\lambda \subset C$ and $P^*(C)=1$, the probability measure P induces a measure m_G on C according to a theorem by Doob (Theorem 1.1, [1]). Specifically this measure is obtained as follows. Let $\mathfrak{F}_C$ be the field of Borel cylinders in C. A Borel cylinder B_C in C is a subset of C of the type

$$B_C = \{x \in C;\ [x(t_1), \cdots, x(t_n)] \in B\} = B_X \cap C$$

where $t_1, \cdots, t_n \in T$, B is a Borel set in the n-dimensional Euclidean space and B_X is as defined previously. If we now define a set function m_G on $\mathfrak{F}_C$ by

$$m_G(B_C) = P(B_X), \tag{4}$$

then m_G is well-defined according to the above quoted theorem by Doob and is in fact a measure on the field $\mathfrak{F}_C$. Finally by means of a Carathéodory extension we have a measure space $(C, \mathfrak{C}^*, m_G)$ in which $\mathfrak{C}^*$ is the σ-field of the Carathéodory measurable sets. Our theorem can now be stated.

THEOREM. *Let* $\{\xi(t), t\in T\}$, $T=[0, 1]$, *be a separable Gaussian process satisfying the conditions* (1), (2), *and* (3). *Let* m_G *be the measure induced on the space* C *of continuous functions* $x(t)$ *defined on* T. *Let* $\lambda>\alpha/2$. *Then for almost every* $x\in C$ *relative to* m_G

$$\limsup_{s\downarrow 0} \frac{|x(t+s)-x(t)|}{s^\lambda} = \infty, \tag{5}$$

$$\limsup_{s\downarrow 0} \frac{|x(t-s)-x(t)|}{s^\lambda} = \infty \tag{6}$$

for almost every $t\in T$ *relative to the Lebesgue measure.*

The proof of this theorem is based on a lemma which we prove in §3. The theorem itself is proved in §4.

3. LEMMA. *Let* $\lambda>\alpha/2$. *Then for every* $t\in(0, 1)$ *there exists a subset of* C, $\Gamma_t\in\mathfrak{C}^*$ *with* $m_G(\Gamma_t)=1$ *such that every* $x\in\Gamma_t$ *satisfies* (5) *and* (6) *at* t.

PROOF. Let $h>0$, $s>0$ and

$$\Gamma_{t,h,s} = \{x \in C;\ |x(t+s) - x(t)| \leqq hs^{\lambda}\}.$$

Then $\Gamma_{t,h,s} \in \mathfrak{F}_C$ and, by (4) and the fact that $\{\xi(t),\ t \in T\}$ is a Gaussian process, we have

$$m_G(\Gamma_{t,h,s}) = \frac{1}{(2\pi)^{1/2}\sigma_{t,t+s}} \int_{|\eta| \leq hs^{\lambda}} \exp\left\{-\frac{\eta^2}{2\sigma^2_{t,t+s}}\right\} d\eta$$

where

$$\sigma^2_{t,t+s} = E\{|\xi(t+s) - \xi(t)|^2\} \geqq as^{\alpha}$$

according to (3). Thus

$$m_G(\Gamma_{t,h,s}) \leqq (2/\pi a)^{1/2} hs^{\lambda - \alpha/2}.$$

Since $\lambda - \alpha/2 > 0$, $\lim_{s \downarrow 0} m_G(\Gamma_{t,h,s}) = 0$. Let $\{s_j\}$ be a sequence of positive numbers such that $\sum_{j=1}^{\infty} s_j^{\lambda - \alpha/2} < \infty$ and consequently $\sum_{j=1}^{\infty} m_G(\Gamma_{t,h,s_j}) < \infty$. By the Borel-Cantelli Theorem, $m_G(\Gamma_{t,h}) = 1$ where $\Gamma_{t,h} = \lim_{j \to \infty} \inf \Gamma^c_{t,h,s_j}$. Thus, from the definition of $\Gamma_{t,h,s}$,

$$\limsup_{s \downarrow 0} \frac{|x(t+s) - x(t)|}{s^{\lambda}} \geqq h \quad \text{for } x \in \Gamma_{t,h}.$$

Let $\Gamma_t^+ = \bigcap_{h=1}^{\infty} \Gamma_{t,h}$. Then $m_G(\Gamma_t^+) = 1$ and

$$\limsup_{s \downarrow 0} \frac{|x(t+s) - x(t)|}{s^{\lambda}} = \infty \quad \text{for } x \in \Gamma_t^+.$$

Similarly, there exists a subset Γ_t^- of C with $m_G(\Gamma_t^-) = 1$ such that

$$\limsup_{s \downarrow 0} \frac{|x(t-s) - x(t)|}{s^{\lambda}} = \infty.$$

We only have to take $\Gamma_t = \Gamma_t^+ \cap \Gamma_t^-$ to complete the proof of the lemma.

4. **Proof of the Theorem.** Consider the product measure $m = m_G \times m_L$ on the product space $C \times T$, where m_L is the Lebesgue measure on T. Let Γ be the subset of $C \times T$ consisting of those elements (x, t) for which (5) and (6) hold. We show that Γ is a measurable subset of $C \times T$ by showing that the two functions of (x, t), $\limsup_{s \downarrow 0} s^{-\lambda}|x(t+s) - x(t)|$ and $\limsup_{s \downarrow 0} s^{-\lambda}|x(t-s) - x(t)|$, are measurable. We assume that each $x \in C$ has been extended beyond T to be constant so that $x(t)$ is continuous in an interval containing T in its interior. Now

$$\limsup_{s \downarrow 0} \frac{|x(t+s) - x(t)|}{s^{\lambda}} = \lim_{k \to \infty} \left\{ \sup_{0 < s < 1/k} \frac{|x(t+s) - x(t)|}{s^{\lambda}} \right\}.$$

Let $\{s_j\}$ be a countable dense subset of $(0, 1/k)$; so that from the continuity of $s^{-\lambda}|x(t+s)-x(t)|$ as a function of s in $(0, 1/k)$, we have

$$\sup_{0<s<1/k} \frac{|x(t+s)-x(t)|}{s^{\lambda}} = \sup_{j} \frac{|x(t+s_j)-x(t)|}{s_j^{\lambda}}.$$

Since each $s_j^{-\lambda}|x(t+s_j)-x(t)|$ is a measurable function of (x, t) on $C\times T$ so is $\limsup_{s\downarrow 0} s^{-\lambda}|x(t+s)-x(t)|$. Similarly,

$$\limsup_{s\downarrow 0} s^{-\lambda}|x(t-s)-x(t)|$$

is measurable, and Γ is measurable.

Now for each $t\in T$, let

$$\Gamma(t) = \{x\in C; (x, t)\in \Gamma\}.$$

For almost every t, $\Gamma(t)$ is a measurable subset of C and for every t, $\Gamma(t)\supset\Gamma_t$ of the Lemma. Since $m_G(\Gamma_t)=1=m_G(C)$, we have

$$\int_0^1 m_G(\Gamma(t))m_L(dt) = \int_0^1 m_G(\Gamma_t)m_L(dt) = 1$$

and hence by Fubini's theorem, $m(\Gamma)=1$ and, furthermore,

$$0 = m(C\times T-\Gamma) = \int_C m_L\{t\in T; (x, t)\notin\Gamma\}m_G(dx).$$

Thus, for almost every $x\in C$,

$$m_L\{t\in T; (x, t)\notin\Gamma\} = 0;$$

that is, for almost every $x\in C$,

$$m_L\{t\in T; (x, t)\in\Gamma\} = 1.$$

This completes the proof of the Theorem.

Bibliography

1. J. L. Doob, *Stochastic processes depending on a continuous parameter*, Trans. Amer. Math. Soc. **42** (1937), 107–140.
2. A. N. Kolmogoroff, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Ergebnisse der Mathematik, No. 3, (1933).
3. M. Loève, *Probability theory*, Princeton Univ. Press, Princeton, N. J., 1963.
4. P. T. Strait, *Sample function regularity for Gaussian processes with the parameter in a Hilbert space*, Ph.D. Thesis, New York Univ., 1965.
5. N. Wiener, *Differential space*, J. Math. and Phys. **2** (1923), 131–174.
6. ———, *Generalized harmonic analysis*, Acta Math. **55** (1930), 117–258.

University of California, Irvine