FOOTNOTE TO THE TITCHMARSH-LINNIK DIVISOR PROBLEM[1]

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Let τ denote the divisor function, and define

$$T_a(x) = \sum_{a<p\leq x} \tau(p-a).$$

By means of his dispersion method, Linnik [1] proved the following

THEOREM. *As* $x\to\infty$, $T_a(x)\sim Ex$, *where*

$$E = \prod_{p\nmid a}\left(1+\frac{1}{p(p-1)}\right)\prod_{p|a}\left(1-\frac{1}{p}\right).$$

The study of $T_a(x)$ was initiated as long ago as 1931 by Titchmarsh [2].

Since the dispersion method is exceedingly complicated, it may be of interest to record that the theorem is a simple consequence of the recent result of Bombieri [3] on the average of the error term in the prime number theorem for arithmetic progressions. For our purpose the following, weaker, form of Bombieri's theorem is sufficient:

(i) *There exists a positive number B such that*

$$\sum_{d\leq x^{1/2}(\log x)^{-B}} \max_{1\leq h<d;(h,d)=1}\left|\pi(x;d,h)-\frac{\operatorname{li} x}{\phi(d)}\right| \ll \frac{x}{\log x}$$

(the notation $\ll$ indicates an inequality with an unspecified constant factor).

We shall require also the following well-known results from elementary number theory (see Prachar [4, p. 44, Satz 4.1] and Estermann [5] respectively):

(ii) *If* $1\leq d<x$, $0\leq h<d$, $(h,d)=1$, *then*

$$\pi(x;d,h) \ll \frac{x}{\phi(d)\log(x/d)}$$

uniformly in x and d.

(iii) $$\sum_{d\leq x;\,(d,a)=1}\frac{1}{\phi(d)} = E\log x + O(1).$$

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[1] Professor Bombieri has pointed out to me that the same observation has already been recorded by G. Rodriguez, Boll. Un. Mat. Ital. in September 1965.

PROOF OF THEOREM. Since

$$\tau(n) = 2 \sum_{d|n;\, d<n^{1/2}} 1 + \theta(n),$$

where $\theta(n) = 1$ if n is a perfect square and is otherwise 0, it follows that

$$\begin{aligned} T_a(x) &= 2 \sum_{a<p\leq x} \sum_{d|(p-a);\, d<(p-a)^{1/2}} 1 + O(x^{1/2}) \\ &= 2 \sum_{d<(x-a)^{1/2}} \sum_{a+d^2<p\leq x;\, p\equiv a \pmod d} 1 + O(x^{1/2}) \\ &= 2 \sum_{d<x^{1/2};\, (d,a)=1} \{\pi(x; d, a) - \pi(a + d^2; d, a)\} + O(x^{1/2}) \\ &= 2 \sum_{d<x^{1/2};\, (d,a)=1} \pi(x; d, a) + O(x/\log x) \end{aligned}$$

by (ii) (with $x = d^2 + a$). Hence, by (i),

$$\begin{aligned} T_a(x) = 2 \operatorname{li} x \sum_{d\leq x^{1/2}(\log x)^{-B};\, (d,a)=1} \frac{1}{\phi(d)} + \sum_{x^{1/2}(\log x)^{-B}<d<x^{1/2};\, (d,a)=1} \pi(x; d, a) \\ + O(x/\log x) = 2 \operatorname{li} x \sum_{d\leq x^{1/2};\, (d,a)=1} \frac{1}{\phi(d)} \\ + O\left(\frac{x}{\log x}\left\{1 + \sum_{x^{1/2}(\log x)^{-B}<d<x^{1/2}} \frac{1}{\phi(d)}\right\}\right) \end{aligned}$$

by (ii); and applying (iii) to both sums on the right we arrive at

$$T_a(x) = Ex + O\left(x \cdot \frac{\log \log x}{\log x}\right),$$

and hence the theorem.

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