A NEW PROOF OF A THEOREM OF P. ERDÖS

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The real valued number-theoretic function[1] $f(n)$ is said to be additive if $f(ab)=f(a)+f(b)$ for $(a, b)=1$[2] and $f(n)$ is completely additive if $f(ab)=f(a)+f(b)$ for any two natural numbers a and b. P. Erdös has proved in [1] the following

THEOREM 1. *For an additive real valued number-theoretic function $f(n)$, $f(n)=c \log n$ is valid with a suitable constant c if one of the following two conditions holds:*

(i) $\lim (f(n+1)-f(n))=0$,[3]

(ii) *$f(n)$ is monotone e.g. $f(n+1) \geqq f(n)$ for all positive integers n.*

The proof of this theorem in [1] is not simple. J. Lambek and L. Moser [2] gave in 1953 a simple proof of Theorem 1 with condition (ii). Later on, A. Rényi [4] simplified the proof when condition (i) holds. In the year preceding Rényi's article, P. Erdös [3, p. 48] stated the following generalization of Theorem 1.

THEOREM 2. *Let $f(n)$ be an additive real valued number-theoretic function. If* $\liminf (f(n+1)-f(n)) \geqq 0$, *then $f(n)=c \log n$ with a suitable constant c.*

This theorem means that if $f(n)=c \log n$ does not hold for any constant c then $f(n+1)-f(n)$ has both positive and negative (finite or infinite) limit points.[4]

Since P. Erdös has not published his proof for Theorem 2, we give here a simple proof for the theorem.[5]

NOTATIONS. We denote by n, k, t natural numbers, by r integers $\geqq 0$, by ϵ an arbitrary positive quantity and by $c(n, \epsilon)$ a quantity which depends only on n and ϵ.

Let $H(\epsilon)$ be the set of the natural numbers n for which $f(n+1)-f(n)<-\epsilon$. If $\liminf (f(n+1)-f(n)) \geqq 0$, the set $H(\epsilon)$ is finite. Put $c(\epsilon)=-\sum_{n \in H(\epsilon)} (f(n+1)-f(n))$.

To prove the theorem we first prove some lemmas.

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[1] I.e., a function defined for all positive integers.

[2] (a, b) denotes the greatest common divisor of a and b.

[3] If this condition holds, then the conclusion is true also for complex valued $f(n)$.

[4] Apply Theorem 2 for $f(n)$ and for $-f(n)$ which is also additive.

[5] P. Erdös and A. Rényi have recently shown that Theorem 3 can be proved by the method of proof given in [4]. Their proof will be published in their forthcoming book on additive and multiplicative number-theoretic functions.

LEMMA 1. *Let S be an arbitrary finite set of natural numbers with r elements. Then*

$$\sum_{n\in S} (f(n+1)-f(n)) \geqq -r\epsilon - c(\epsilon).$$

PROOF. The lemma follows trivially from the definition of $c(\epsilon)$.

LEMMA 2.

$$\text{(A)} \qquad f(n^k) - kf(n) \geqq c_1(n,\epsilon) - kn\epsilon,$$

$$\text{(B)} \qquad f(n^k) - kf(n) \leqq c_2(n,\epsilon) + kn\epsilon.$$

PROOF. (A) By Lemma 1 a trivial calculation shows:

$$\begin{aligned} f(n^k)-kf(n) &= (f(n^k)-f(n^k-1)) \\ &\quad + \sum_{r=2}^{k} (f(n^r-1)-f(n^{r-1}-1)-f(n))+(f(n-1)-f(n)) \\ &= (f(n^k)-f(n^k-1))+\sum_{r=2}^{k} (f(n^r-1)-f(n^r-n))+(f(n-1)-f(n)) \\ &\geqq -kn\epsilon-c(\epsilon)+f(n-1)-f(n)=c_1(n,\epsilon)-kn\epsilon. \end{aligned}$$

Q.E.D.

(B) In the same way we get by Lemma 1

$$\begin{aligned} f(n^k)-kf(n) &= (f(n^k)-f(n^k+1)) \\ &\quad + \sum_{r=2}^{k} (f(n^r+1)-f(n^{r-1}+1)-f(n))+(f(n+1)-f(n)) \\ &= (f(n^k)-f(n^k+1))+\sum_{r=2}^{k} (f(n^r+1)-f(n^r+n))+(f(n+1))-f(n)) \\ &\geqq kn\epsilon+c(\epsilon)+f(n+1)-f(n)=c_2(n,\epsilon)+kn\epsilon. \end{aligned}$$

Q.E.D.

COROLLARY 2.1.
(A) $f(n^t)=tf(n)$,
(B) *$f(n)$ is completely additive.*

PROOF. The statement (B) is a trivial following of the statement (A) hence we prove here only (A).

By Lemma 2

$$\left| f(n^k) - kf(n) \right| \leqq c_3(n,\epsilon) + kn\epsilon,$$

i.e., we have

$$\limsup_{k\to+\infty} \left| \frac{f(n^k)}{k} - f(n) \right| \leqq n\epsilon \qquad \text{for every } \epsilon > 0,$$

i.e.,

$$\lim_{k\to+\infty}\left|\frac{f(n^k)}{k}-f(n)\right|=0,\quad \text{i.e.,}\quad \lim_{k\to+\infty}\frac{f(n^k)}{k}=f(n).$$

Having applied this result we get

$$f(n^t)=\lim_{k\to+\infty}\frac{f(n^{tk})}{k}=t\lim_{k\to+\infty}\frac{f(n^k)}{k}=tf(n).\qquad \text{Q.E.D.}$$

LEMMA 3. *Let* $2^k\leqq n<2^{k+1}$. *Then*

(A) $$f(n)\geqq -k\epsilon-c(\epsilon)+kf(2),$$

(B) $$f(n)\leqq (k+1)\epsilon+c(\epsilon)+(k+1)f(2).$$

PROOF. (A) Let $x_0=n$. Furthermore, if x_r has been defined, let x_r' be the even of the numbers x_r and x_r-1 and let $x_{r+1}=x_r'/2$. It is easy to see that $x_k=1$. Moreover since $f(x_k)=f(1)=0$ for each additive function (we apply Corollary 2.1 (B) and Lemma 1),

$$f(n)=f(x_0)-f(x_k)=\sum_{r=0}^{k-1}(f(x_r)-f(x_r'))+\sum_{r=0}^{k-1}(f(x_r')-f(x_{r+1}))$$

$$=\sum_{r=0}^{k-1}(f(x_r)-f(x_r'))+\sum_{r=0}^{k-1}f(2)\geqq -k\epsilon-c(\epsilon)+kf(2).\quad \text{Q.E.D.}$$

(B) Let $y_0=n$. Furthermore, if y_r has been defined let y_r' be the even of the numbers y_r and y_r+1 and let $y_{r+1}=y_r'/2$. Then $y_{k+1}=1$. Moreover since $f(y_{k+1})=0$ (we apply Corollary 2.1 (B) and Lemma 1):

$$f(n)=f(y_0)-f(y_{k+1})=\sum_{r=0}^{k}(f(y_r)-f(y_r'))+\sum_{r=0}^{k}(f(y_r')-f(y_{r+1}))$$

$$=\sum_{r=0}^{k}(f(y_r)-f(y_r'))+\sum_{r=0}^{k}f(2)\leqq (k+1)\epsilon+c(\epsilon)+(k+1)f(2).$$

Q.E.D.

COROLLARY 3.1. $|f(n)-f(2)\log_2 n|<2\epsilon\log_2 n+c(\epsilon)+2|f(2)|$.

PROOF. Lemma 3 shows that $|f(n)-kf(2)|<2k\epsilon+c(\epsilon)\leqq|f(2)|$. Since $k\leqq\log_2 n<k+1$ we get the statement of the corollary.

Now we can prove Theorem 2 as follows.

Replacing n by n^t in Corollary 3.1 and dividing by t we obtain (having applied also the statement of Corollary 2.1 (A))

$$|f(n)-f(2)\log_2 n|<2\epsilon\log_2 n+\frac{c(\epsilon)+2|f(2)|}{t}.$$

If $t\to+\infty$ we get

$$|f(n) - f(2)\log_2 n| \leqq 2\epsilon \log_2 n \qquad \text{for every } \epsilon > 0$$

which shows that $f(n)=f(2)\log_2 n$. The theorem is proved.

RELATED PROBLEMS.[6] Here we list some unsolved problems.

Let $f(n+1)-f(n)$ be bounded. Does it follow that $f(n)=c\log n+g(n)$ wherein $g(n)$ is bounded?

Assume that $f(n+1)-f(n)\to 0$ except for the n-s which form a sequence of density 0. Does it follow that $f(n)=c\log n$? What can be asserted if $f(n+1)\geqq f(n)$ or lim inf $f(n+1)-f(n)\geqq 0$ after neglecting a sequence of density 0?

Assume that

$$\sum_{r=1}^{n} |f(r+1) - f(r)| = o(n).$$

Does it follow that $f(n)=c\log n$?

REFERENCES

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[6] These problems are due to P. Erdös and I published them with his permission.