for \( n = k + 1 \) if it holds for \( n = k \). Hence, by induction, it holds for every \( n \).

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**References**


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**SOME GENERALIZATIONS OF OPIAL’S INEQUALITY**

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The inequality \( \int_0^a |uu'| \leq a/2 \int_0^a |u'|^2 \) which is valid for absolutely continuous \( u \) with \( u(0) = 0 \) has received successively simpler proofs by Opial, [5], Olech [4], Beesack [1], Levinson [2], Pederson [6], and Mallows [3]. It is the purpose of this paper to use the method of Olech to obtain some more general inequalities.

**Theorem 1.** Let \( u \) be absolutely continuous on \((a, b)\) with \( u(a) = 0 \), where \(-\infty \leq a < b < \infty\). Let \( f(t) \) be a continuous, complex function defined for all \( t \) in the range of \( u \) and for all real \( t \) of the form \( t(s) = \int_0^s |u'(x)| \, dx \). Suppose that \( |f(t)| \leq f(|t|) \), for all \( t \), and that \( f(t_1) \leq f(t_2) \) for \( 0 \leq t_1 \leq t_2 \). Let \( r \) be positive, continuous and in \( L^{1-q}[a, b] \), where \( 1/p + 1/q = 1, p > 1 \). Let \( F(s) = \int_0^s f(x) \, dx, s > 0 \). Then

\[
\int_a^b |f(u)u'| \, dx \leq F \left[ \left( \int_a^b r^{1-q} \right)^{1/q} \left( \int_a^b r \, u'|^p \right)^{1/p} \right]
\]

with equality iff \( u(x) = A f_0^{b} r^{1-q} \). The same result (but with equality for \( u(x) = \int_0^b r^{1-q} \)) holds if \( u(b) = 0 \) and \(-\infty < a < b \leq \infty\).

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Proof. Let \( z(x) = \int_a^x |u'| \),
\[
\int_a^b |f(u(x))u'(x)| \, dx = \int_a^b |f \left( \int_a^x u' \right) u'(x)| \, dx
\]
\[
\leq \int_a^b f \left( \left| \int_a^x u' \right| \right) |u'(x)| \, dx
\]
\[
= \int_a^b f(z)z^p \, dx = F(z(b)),
\]
\[
z(b) = \int_a^b z' = \int_a^b r^{-1/p}r^{-1/p} = \left( \int_a^b r^{-q} \right)^{1/q} \left( \int_a^b r^p \right)^{1/p}.
\]
The result follows with the observation that \( F \) is nondecreasing.

Example. Let \( f(t) = t^{p-1}, p > 1, u(a) = 0, -\infty < a < b < \infty \), then
\[
\int_a^b |u^{-1}u'| \leq (1/p) \left( \int_a^b r^{-q} \right)^{p-1} \int_a^b r \, |u'|^p
\]
with equality iff \( u(x) = A \int_a^x r^{1-q} \). For \( p = 2 \), we obtain essentially Bee-
sack's generalization \([1]\).

Example. Let \( f(t) = \sum_{n=0}^{\infty} a_n t^n \) be an absolutely convergent power series with radius of convergence \( R \). Let \( F(s) = \int_a^b \sum_{n=0}^{\infty} |a_n| x^n dx \). If \( \int_a^b |u'| < R \), then the theorem is true with this choice for \( f \) since
\[
\int_a^b f(u)u' \leq \int_a^b \sum_{n=0}^{\infty} |a_n| |u| |u'| \text{ and the function } g(t) = \sum |a_n| t^n
\]
has the properties \( g(t) \leq g(|t|) \), \( g(t_1) \leq g(t_2) \) when \( 0 \leq t_1 \leq t_2 \).

Corollary. If \( F(t+s) \geq F(t) + F(s) \) for \( t, s \geq 0 \) and if \( u(a) = u(b) = 0 \) for \( -\infty < a < b < \infty \), then
\[ F(\xi) = F(\xi) + F(\xi) \text{ where } \lambda = (\int_a^b r^{1-q})^{p-1} = (\int_a^b r^{1-q})^{p-1} \text{ and } \xi \text{ is the uniquely determined point where the two integrals are equal. Equality holds iff}
\]
\[
u(x) = A \int_a^x r^{1-q}, \quad a \leq x \leq \xi
\]
\[
= A \int_\xi^b r^{1-q}, \quad \xi \leq x \leq b.
\]

Proof. We have
\[
\int_a^\xi |f(u)u'| \leq F \left[ \left( \int_a^\xi r^{1-q} \right)^{1/q} \left( \int_a^\xi r \, |u'|^p \right)^{1/p} \right]
\]
and
\[
\int_{\xi}^{b} |f(u)u'| \leq F \left[ \left( \int_{\xi}^{b} r^{1-q} \right)^{1/q} \left( \int_{\xi}^{b} r \left| u' \right|^p \right)^{1/p} \right].
\]

The result follows by adding, using the fact that \( F(s+t) \geq F(s) + F(t) \), and noting the choice of \( \xi \).

**Example.** For \( r = 1, a \) and \( b \) finite, and \( f(t) = t^{p-1} \) for \( p > 1 \),
\[
\int_{a}^{b} |u^{p-1}u'| \leq (1/p) \left[ (b-a)/2 \right]^{p-1} \int_{a}^{b} |u'|^p
\]
with equality for
\[
u(x) = \begin{cases} A(x-a); & a \leq x \leq (a+b)/2, \\ A(b-x); & (a+b)/2 \leq x \leq b. \end{cases}
\]

For \( p=2 \) and \( a=0 \) we obtain Opial's inequality.

**Example.** Let \( f(t) = t^{p-1} \) for \( p > 1 \) and \( u = v^{1/p} \), then
\[
\max_{t} |v(t)| \leq 1/2 \int_{a}^{b} |v'| \leq (2p)^{-p} (b-a)^{p-1} \int_{a}^{b} |v|^{1-p} |v'|^p
\]

**Theorem 2.** Let \( u, f, \) and \( r \) be as in Theorem 1. If \( p < 1, 1/p + 1/q = 1, u(a) = 0 \), \( -\infty \leq a < b \leq \infty \), then
\[
\int_{a}^{b} \left| u'/f(u) \right| \, dx \geq G \left[ \left( \int_{a}^{b} r^{1-q} \right)^{1/q} \left( \int_{a}^{b} r \left| u' \right|^p \right)^{1/p} \right],
\]
where \( G(s) = \int_{0}^{s} 1/f(x) \, dx \).

Equality holds iff \( u(x) = \int_{a}^{x} r^{1-q} \). If \( u(b) = 0 \), \( -\infty < a < b \leq \infty \), the same result holds.

**Proof.**
\[
\int_{a}^{b} \left| u'/f(u) \right| = \int_{a}^{b} \left| u' \right| / \left| f \left( \int_{a}^{b} u' \right) \right|
\]
\[
geq \int_{a}^{b} \left| u' \right| / f \left( \int_{a}^{b} u' \right)
\]
\[
= \int_{a}^{b} z'/f(z) = G(z(b))
\]
\[
z(b) = \int_{a}^{b} z' = \int_{a}^{b} r^{-1/p} r^1/p z' \geq \left( \int_{a}^{b} r^{1-q} \right)^{1/q} \left( \int_{a}^{b} r \left| u' \right|^p \right)^{1/p}.
\]

The result follows with the observation that \( G \) is nondecreasing.
Example. For $0 < p < 1$, $f(t) = t^{p-1}$;

$$\int_a^b |u^{p-1}u'| \geq (1/p) \left( \int_a^b r^{1-q} \right)^{p-1} \left( \int_a^b r \ |u'|^p \right).$$

Taking $r = 1$, $p = 1/2$, $u = v^2$, we obtain

$$\int_a^b |vv'|^{1/2} \leq 2^{-1/2}(b-a)^{1/2} \int_a^b |v'|.$$

The next theorem is a generalization of a different sort which can easily be proven by the methods used above.

**Theorem 3.** If $u$ and $v$ are absolutely continuous for $-\infty \leq a < b < \infty$ and if $u(a) = v(a) = 0$, then

$$\int_a^b |uv'| + |vu'| \leq \left[ \int_a^b r^{-2} \int_a^b s^{-2} \int_a^b r^2 |u'|^2 \int_a^b s^2 |v'|^2 \right]^{1/2}$$

with equality iff $u(x) = A \int_a^x r^{-2}$ and $v(x) = B \int_a^x s^{-2}$.

**References**


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