for $n=k+1$ if it holds for $n=k$. Hence, by induction, it holds for every n.

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SOME GENERALIZATIONS OF OPIAL'S INEQUALITY

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The inequality $\int_0^a |uu'| \leqq a/2\int_0^a |u'|^2$ which is valid for absolutely continuous u with $u(0)=0$ has received successively simpler proofs by Opial, [5], Olech [4], Beesack [1], Levinson [2], Pederson [6], and Mallows [3]. It is the purpose of this paper to use the method of Olech to obtain some more general inequalities.

THEOREM 1. *Let u be absolutely continuous on (a, b) with $u(a)=0$, where $-\infty \leqq a<b<\infty$. Let $f(t)$ be a continuous, complex function defined for all t in the range of u and for all real t of the form $t(s) =\int_a^s |u'(x)| dx$. Suppose that $|f(t)| \leqq f(|t|)$, for all t, and that $f(t_1) \leqq f(t_2)$ for $0\leqq t_1\leqq t_2$. Let r be positive, continuous and in $L^{1-q}[a, b]$, where $1/p+1/q=1$, $p>1$. Let $F(s)=\int_0^s f(x)dx$, $s>0$. Then*

$$\int_a^b |f(u)u'| \, dx \leqq F\left[\left(\int_a^b r^{1-q}\right)^{1/q}\left(\int_a^b r|u'|^p\right)^{1/p}\right]$$

with equality iff $u(x)=A\int_a^x r^{1-q}$. The same result (but with equality for $u(x)=\int_x^b r^{1-q}$) holds if $u(b)=0$ and $-\infty<a<b\leqq\infty$.

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PROOF. Let $z(x)=\int_a^x|u'|$,

$$\int_a^b |f(u(x))u'(x)|\,dx = \int_a^b \left|f\left(\int_a^x u'\right)u'(x)\right|dx$$

$$\leqq \int_a^b f\left(\left|\int_a^x u'\right|\right)|u'(x)|\,dx$$

$$\leqq \int_a^b f\left(\left|\int_a^x |u'|\right)\right)|u'(x)|\,dx$$

$$= \int_a^b f(z)z'dx = F(z(b)),$$

$$z(b) = \int_a^b z' = \int_a^b r^{-1/p}r^{1/p}z' \leqq \left(\int_a^b r^{1-q}\right)^{1/q}\left(\int_a^b rz'^p\right)^{1/p}.$$

The result follows with the observation that F is nondecreasing.

EXAMPLE. Let $f(t)=t^{p-1}$, $p>1$, $u(a)=0$, $-\infty\leqq a<b<\infty$, then

$$\int_a^b |u^{p-1}u'| \leqq (1/p)\left(\int_a^b r^{1-q}\right)^{p-1}\int_a^b r|u'|^p$$

with equality iff $u(x)=A\int_a^x r^{1-q}$. For $p=2$, we obtain essentially Beesack's generalization [1].

EXAMPLE. Let $f(t)=\sum_{n=0}^\infty a_nt^n$ be an absolutely convergent power series with radius of convergence R. Let $F(s)=\int_0^s\sum_{n=0}^\infty |a_n|x^ndx$. If $\int_a^b|u'|<R$, then the theorem is true with this choice for f since $\int_a^b|f(u)u'|\leqq\int_a^b\sum_{n=0}^\infty|a_n|\,|u|^n|u'|$ and the function $g(t)=\sum|a_n|t^n$ has the properties $|g(t)|\leqq g(|t|)$, $g(t_1)\leqq g(t_2)$ when $0\leqq t_1\leqq t_2$.

COROLLARY. *If $F(t+s)\geqq F(t)+F(s)$ for $t, s\geqq 0$ and if $u(a)=u(b)=0$ for $-\infty\leqq a<b\leqq\infty$, then $\int_a^b|f(u)u'|\leqq F[\lambda(\int_a^b r|u'|^p)^{1/p}]$ where $\lambda=(\int_a^\xi r^{1-q})^{p-1}=(\int_\xi^b r^{1-q})^{p-1}$ and ξ is the uniquely determined point where the two integrals are equal. Equality holds iff*

$$u(x) = A\int_a^x r^{1-q};\qquad a\leqq x\leqq \xi$$

$$= A\int_x^b r^{1-q};\qquad \xi\leqq x\leqq b.$$

PROOF. We have

$$\int_a^\xi |f(u)u'| \leqq F\left[\left(\int_a^\xi r^{1-q}\right)^{1/q}\left(\int_a^\xi r|u'|^p\right)^{1/p}\right]$$

and

$$\int_\xi^b |f(u)u'| \leqq F\left[\left(\int_\xi^b r^{1-q}\right)^{1/q}\left(\int_\xi^b r|u'|^p\right)^{1/p}\right].$$

The result follows by adding, using the fact that $F(s+t) \geqq F(s)+F(t)$, and noting the choice of ξ.

EXAMPLE. For $r=1$, a and b finite, and $f(t)=t^{p-1}$ for $p>1$ $\int_a^b |u^{p-1}u'| \leqq (1/p)[(b-a)/2]^{p-1}\int_a^b |u'|^p$ with equality for

$$\begin{aligned} u(x) &= A(x-a);\ a \leqq x \leqq (a+b)/2, \\ &= A(b-x);\ (a+b)/2 \leqq x \leqq b. \end{aligned}$$

For $p=2$ and $a=0$ we obtain Opial's inequality.

EXAMPLE. Let $f(t)=t^{p-1}$ for $p>1$ and $u=v^{1/p}$, then

$$\max_t |v(t)| \leqq 1/2\int_a^b |v'| \leqq (2p)^{-p}(b-a)^{p-1}\int_a^b |v|^{1-p}|v'|^p$$

THEOREM 2. *Let u, f, and r be as in Theorem* 1. *If $p<1$, $1/p+1/q=1$, $u(a)=0$, $-\infty \leqq a<b<\infty$, then*

$$\int_a^b |u'/f(u)|\,dx \geqq G\left[\left(\int_a^b r^{1-q}\right)^{1/q}\left(\int_a^b r|u'|^p\right)^{1/p}\right],$$

$$\textit{where } G(s) = \int_0^s 1/f(x)\,dx.$$

Equality holds iff $u(x)=\int_a^x r^{1-q}$. If $u(b)=0$, $-\infty<a<b\leqq\infty$, the same result holds.

PROOF.

$$\begin{aligned} \int_a^b |u'/f(u)| &= \int_a^b |u'| \Big/ \left|f\left(\int_a^b u'\right)\right| \\ &\geqq \int_a^b |u'| \Big/ f\left(\int_a^b |u'|\right) \\ &= \int_a^b z'/f(z) = G(z(b)) \end{aligned}$$

$$z(b) = \int_a^b z' = \int_a^b r^{-1/p}r^{1/p}z' \geqq \left(\int_a^b r^{1-q}\right)^{1/q}\left(\int_a^b r|u'|^p\right)^{1/p}.$$

The result follows with the observation that G is nondecreasing.

EXAMPLE. For $0<p<1$, $f(t)=t^{p-1}$;

$$\int_a^b |u^{p-1}u'| \geqq (1/p)\left(\int_a^b r^{1-q}\right)^{p-1}\left(\int_a^b r\,|u'|^p\right).$$

Taking $r=1$, $p=1/2$, $u=v^2$, we obtain

$$\int_a^b |vv'|^{1/2} \leqq 2^{-1/2}(b-a)^{1/2}\int_a^b |v'|.$$

The next theorem is a generalization of a different sort which can easily be proven by the methods used above.

THEOREM 3. *If u and v are absolutely continuous for $-\infty \leqq a<b<\infty$ and if $u(a)=v(a)=0$, then*

$$\int_a^b |uv'| + |vu'| \leqq \left[\int_a^b r^{-2}\int_a^b s^{-2}\int_a^b r^2|u'|^2\int_a^b s^2|v'|^2\right]^{1/2}$$

with equality iff $u(x)=A\int_a^b r^{-2}$ and $v(x)=B\int_a^b s^{-2}$.

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