ON A CLASS OF POLYNOMIALS OBTAINED FROM GENERALIZED HUMBERT POLYNOMIALS

DAVID ZEITLIN

1. **Introduction.** Recently, Gould [1] defined a generalized Humbert polynomial, $P_n(m, x, y, p, C)$, of degree n in x, by the series expansion

$$(1.1) \qquad (C - mxt + yt^m)^p = \sum_{n=0}^{\infty} t^n P_n(m, x, y, p, C),$$

where $m \geqq 1$ is an integer and the other parameters, C, x, y, and p, are unrestricted in general. By a proper choice of parameters, one obtains from (1.1) the generating function for such polynomials as Gegenbauer, Legendre, Humbert ($P_n(m, x, 1, -p, 1)$), and Chebyshev polynomials, $U_n(x)$, of the second kind, as important special cases. An extensive list of references is given in [1].

The aim of this paper is to give the important properties of the polynomials $R_n(x, m, C/y)$ which we define by the limit

$$(1.2) \qquad \begin{aligned} &\lim_{p \to 0} P_n(m, x, y, p, C)/p \\ &\quad = -(m/n)(C/y)^{-n/2} \cdot R_n(mx/(2(Cy)^{1/2}), m, C/y) \qquad (n \neq 0). \end{aligned}$$

Noting that for $k > 0$, $P_n(2, x, 1, -k, 1) = C_n^{(k)}(x)$, the Gegenbauer polynomial, and recalling [2, p. 82, (4.7.8)] that $\lim_{k \to 0} C_n^{(k)}(x)/k = (2/n)T_n(x)$, $n \neq 0$, where $T_n(x)$ is the Chebyshev polynomial of the first kind, it follows from (1.2) that $R_n(x, 2, 1) \equiv T_n(x)$, which is indeed the case (see (1.4)).

We shall cite only those results of [1] which are needed to make this paper self-contained. In [1, p. 699, (2.1)] it is shown that

$$(1.3) \qquad \begin{aligned} &P_n(m, x, y, p, C) \\ &\quad = \sum_{k=0}^{[n/m]} \binom{p}{k} \binom{p-k}{n-mk} C^{p-n+(m-1)k} y^k (-mx)^{n-mk}. \end{aligned}$$

Applying (1.2) to (1.3) and noting that

$$\binom{p}{k} = p(p-1) \cdots (p-k+1)/k!,$$

Received by the editors May 3, 1966.

we obtain

$$\lim_{p\to 0}\binom{p}{k}\binom{p-k}{n-mk}\Big/ p = (-1)^{n-mk+k-1}\frac{[n-1-(m-1)k]!}{k!(n-mk)!};$$

and hence, after simplification, with $b=C/y$, that

$$(1.4)\qquad R_n(x,m,b) = \frac{n}{m}\sum_{k=0}^{[n/m]}(-1)^k\frac{[n-1-(m-1)k]!}{k!(n-mk)!}b^{(m-2)k/2}(2x)^{n-mk} \qquad (n=1,2,\cdots).$$

It is convenient to define $R_0(x, m, b)\equiv 1$. We note that $R_n(x, 2, b) = T_n(x)$, and $R_n(x, 1, b)=(2x-b^{-\frac{1}{2}})^n$.

The inverse relation to (1.4) is given by

$$(1.5)\qquad (2x)^n = m\sum_{k=0}^{[n/m]}\binom{n}{k}b^{(m-2)k/2}R_{n-mk}(x,m,b),$$

which was obtained by applying (1.2) to the inverse relation of (1.3) (see [**1**, p. 707, (6.2)])

$$(1.6)\qquad \binom{p}{n}(-mx)^n = \sum_{k=0}^{[n/m]}(-1)^k\binom{p-n+k}{k}\frac{p+mk-n}{p+k-n}\cdot C^{n-k-p}y^kP_{n-mk}(m,x,y,p,C).$$

For $m=2$, (1.5) reduces to the known relation for $T_n(x)$ (see, e.g., [**3**, p. 223, Problem 4b]). In the sequel, we shall write $R_n(x, m, b)\equiv R_n$ and $P_n(m, x, y, p, C)\equiv P_n$.

2. **Properties of the polynomials.** R_n satisfies a linear difference equation of order m with constant coefficients

$$(2.1)\qquad R_{n+m} - 2xR_{n+m-1} + b^{(m-2)/2}R_n = 0 \qquad (n=0,1,\cdots),$$

and from (1.4), $R_n=(2x)^n/m$, $n=1, 2, \cdots, m-1$, and $R_0=1$. For $m=2$, (2.1) is the difference equation satisfied by Chebyshev polynomials. We obtained (2.1) by applying (1.2) to

$$(2.2)\qquad CnP_n - m(n-1-p)xP_{n-1} + (n-m-mp)yP_{n-m} = 0 \qquad (n\geqq m\geqq 1),$$

(see [**1**, p. 700, (2.3)]). Applying a well-known result (see [**4**, p. 300]) for generating functions, i.e., if $Y_0, Y_1, \cdots, Y_N$ are arbitrary real numbers and Y_n, $n=0, 1, \cdots$, satisfies a homogeneous, linear difference equation of order $N+1$ with real, constant coefficients

$$\sum_{r=0}^{N+1} a_r Y_{n+N+1-r} = 0 \qquad (a_0 a_{N+1} \neq 0),$$

then the generating function of Y_n is given by

$$\sum_{n=0}^{\infty} Y_n t^n = \sum_{j=0}^{N} \left(\sum_{r=0}^{j} a_r Y_{j-r} \right) t^j \Big/ \sum_{r=0}^{N+1} a_r t^r \quad (N = 0, 1, \cdots), \tag{2.3}$$

we obtain from (2.3) with $N=m-1$, $a_0=1$, $a_1=-2x$, $a_r=0$, $r=2, 3, \cdots, m-1$, and $a_m=b^{(m-2)/2}$, the generating function for R_n,

$$\frac{1 + (2(1-m)/m)xt}{1 - 2xt + b^{(m-2)/2}t^m} = \sum_{n=0}^{\infty} t^n R_n(x, m, b). \tag{2.4}$$

For $m=2$, (2.4) reduces to the generating function for $T_n(x)$.

It is now possible to express R_n in terms of P_n. In (1.1), set $p=-1$, $C=1$, and replace x by $2x/m$ to obtain

$$1/(1 - 2xt + yt^m) = \sum_{n=0}^{\infty} t^n P_n(m, 2x/m, y, -1, 1). \tag{2.5}$$

Set $y = b^{(m-2)/2}$ in (2.5) and then multiply both sides by $1+(2(1-m)/m)xt$ to obtain, noting (2.4),

$$\begin{aligned} R_n(x, m, b) &= P_n(m, 2x/m, b^{(m-2)/2}, -1, 1) \\ &\quad + (2(1-m)/m)xP_{n-1}(m, 2x/m, b^{(m-2)/2}, -1, 1) \\ &\qquad (n = 1, 2, \cdots). \end{aligned} \tag{2.6}$$

For $m=2$, (2.6) gives

$$T_n(x) = P_n(2, x, 1, -1, 1) - xP_{n-1}(2, x, 1, -1, 1) \qquad (n = 1, 2, \cdots). \tag{2.7}$$

Conversely, we have

$$P_n(m, 2x/m, b^{(m-2)/2}, -1, 1) = \sum_{j=0}^{n} (2x(m-1)/m)^{n-j} R_j(x, m, b); \tag{2.8}$$

and for $m=2$, (2.8) gives

$$C_n^{(1)}(x) = U_n(x) = \sum_{j=0}^{n} x^{n-j} T_j(x). \tag{2.9}$$

In [1, p. 702, (3.5)], it was shown that

$$D_x^k P_{n+k}(m, x, y, p, C) = k!(-m)^k \binom{p}{k} P_n(m, x, y, p-k, C) \tag{2.10}$$

$$(D_x^k = d^k/dx^k).$$

Applying (1.2) to (2.10), we obtain, after simplification,

$$\begin{aligned} mD_x^k R_{n+k}(x, m, C/y) &= (k-1)!(n+k)2^k(C/y)^{n/2} \\ &\cdot C^k P_n(m, 2x(Cy)^{1/2}/m, y, -k, C) \qquad (k = 1, 2, \cdots). \end{aligned} \tag{2.11}$$

Since $P_n(2, x, 1, -k, 1) = C_n^{(k)}(x)$, the Gegenbauer polynomial, we obtain from (2.11)

$$D_x^k T_{n+k}(x) = (k-1)!(n+k)2^{k-1} \cdot C_n^{(k)}(x) \qquad (k = 1, 2, \cdots). \tag{2.12}$$

Using (1.3), we note that

$$C^k P_n(m, 2x(Cy)^{1/2}/m, y, -k, C) = P_n(m, 2x(y/C)^{1/2}/m, y/C, -k, 1),$$

and thus, (2.11), with $b = C/y$, becomes

$$\begin{aligned} &mD_x^k R_{n+k}(x, m, b) \\ &= (k-1)!(n+k)2^k b^{n/2} P_n(m, 2xb^{-1/2}/m, 1/b, -k, 1) \quad (k = 1, 2, \cdots); \end{aligned} \tag{2.13}$$

and for $n = 0$, (2.13) gives

$$D_x^k R_k(x, m, b) = k!2^k/m \qquad (k = 1, 2, \cdots). \tag{2.14}$$

The polynomials R_n also satisfy the following recurrence relations, where $R_n' = D_x R_n$:

$$\begin{aligned} R_n' &= (x/n)R_n' \\ &\quad - ((m/2)/(n-m+1))b^{(m-2)/2}R_{n-m+1}', \end{aligned} \tag{2.15}$$

$$\begin{aligned} 2R_n &= (1/(n+1))R_{n+1}' \\ &\quad - ((m-1)/(n-m+1))b^{(m-2)/2}R_{n-m+1}', \end{aligned} \tag{2.16}$$

$$R_n = ((m/2)/(n+1))R_{n+1}' - ((m-1)x/n)R_n', \tag{2.17}$$

$$\begin{aligned} (2(m-1)x^2/n)R_n' &= 2(m-1)xR_n \\ &\quad + ((m^2/2)/(n-m+2))b^{(m-2)/2}R_{n-m+2}' \\ &\quad - mb^{(m-2)/2}R_{n-m+1}. \end{aligned} \tag{2.18}$$

Relations (2.15), $\cdots$, (2.18) were obtained, respectively, by applying (1.2) to [**1**, p. 700, (2.5), (2.6), (2.7), (2.8)].

3. **An expansion theorem.** In [**1**, pp. 709–710], a theorem was given for the expansion of a polynomial of degree s in x as a linear combination of generalized Humbert polynomials. A similar theorem is now given for an expansion in terms of the derived polynomials R_n.

THEOREM 1. *Let $f(x)=\sum_{n=0}^{s} A_n x^n$ be an arbitrary polynomial of degree s in x. Then $f(x)$ may be expressed in terms of the polynomials R_n by the formula*

$$f(x) = K_0/m + \sum_{n=1}^{s} K_n R_n(x, m, b), \tag{3.1}$$

where

$$K_n = m \sum_{k=0}^{[(s-n)/m]} \binom{n+mk}{k} 2^{-(n+mk)} b^{(m-2)k/2} A_{n+mk} \qquad (n = 0, 1, \cdots, s). \tag{3.2}$$

PROOF. In [**1**, p. 709], there is stated the identity

$$\sum_{n=0}^{s} \sum_{k=0}^{[n/m]} B_{n,k} = \sum_{n=0}^{s} \sum_{k=0}^{[(s-n)/m]} B_{n+mk,k}. \tag{3.3}$$

If we substitute for x^n (as given by (1.5)) into $\sum_{n=0}^{s} A_n x^n$, we obtain our result (3.1) and (3.2) by an application of (3.3).

4. **Additional results.** In [**5**, p. 135], it was shown that if the roots of the characteristic equation for the difference equation

$$Y_{n+m} = qY_{n+r} + pY_n \qquad (pq \neq 0,\ 1 \leqq r < m,\ n = 0, 1, \cdots), \tag{4.1}$$

are distinct, then

$$Y_{nm} = \sum_{k=0}^{n} \binom{n}{k} p^{n-k} q^k Y_{kr} \qquad (n = 0, 1, \cdots). \tag{4.2}$$

Since the characteristic equation of (2.1) is

$$z^m - 2xz^{m-1} + b^{(m-2)/2} = 0, \tag{4.3}$$

then (4.2), with $Y_n \equiv R_n$ and $r = m-1$, gives

$$R_{nm}(x, m, b) = \sum_{k=0}^{n} \binom{n}{k} (-b^{(m-2)/2})^{n-k} (2x)^k R_{k(m-1)}(x, m, b) \qquad (n = 0, 1, \cdots), \tag{4.4}$$

which is valid provided the roots of (4.3) are distinct. For $m=2$, (4.4) gives

$$(4.5)\qquad (-1)^n T_{2n}(x) = \sum_{k=0}^{n} \binom{n}{k} (-2x)^k T_k(x) \qquad (x \neq \pm 1, n = 0, 1, \cdots).$$

Since $U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x)$, we also have

$$(4.6)\qquad (-1)^n U_{2n}(x) = \sum_{k=0}^{n} \binom{n}{k} (-2x)^k U_k(x) \qquad (x \neq \pm 1, n = 0, 1, \cdots).$$

In [**6**, p. 253, (5.1), (5.2)], it was shown that

$$(4.7)\qquad V_n = 2a^{n/2}T_n(d/(2(a)^{1/2})) \qquad (n = 0, 1, \cdots).$$

$$(4.8)\qquad Z_{n+1} = a^{n/2}U_n(d/(2(a)^{1/2})) \qquad (n = 0, 1, \cdots),$$

where Z_n and V_n are Lucas functions which satisfy

$$(4.9)\qquad W_{n+2} = dW_{n+1} - aW_n, \quad d^2 - 4a \neq 0, \qquad (n = 0, 1, \cdots),$$

with $d \neq 0$ and $a \neq 0$ as arbitrary real numbers; and $Z_n \equiv W_n$ if $W_0 = 0$ and $W_1 = 1$, and $V_n \equiv W_n$ if $W_0 = 2$ and $W_1 = d$. Applying (4.7) to (4.5) and (4.8) to (4.6), we obtain, respectively,

$$(4.10)\qquad (-1)^n V_{2n} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a^{n-k} d^k V_k \qquad (n = 0, 1, \cdots),$$

$$(4.11)\qquad (-1)^n Z_{2n+1} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a^{n-k} d^k Z_{k+1} \qquad (n = 0, 1, \cdots).$$

An application of (4.2) to (4.9) gives

$$(4.12)\qquad W_{2n} = \sum_{k=0}^{n} \binom{n}{k} (-a)^{n-k} d^k W_k \qquad (n = 0, 1, \cdots).$$

If we apply the inverse relation pair identity

$$(4.13)\qquad A_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} B_k, \quad B_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} A_k,$$

to (4.12) and (4.11), we obtain, respectively,

$$(4.14)\qquad d^n W_n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} W_{2k} \qquad (n = 0, 1, \cdots).$$

$$(4.15)\qquad d^n Z_{n+1} = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} Z_{2k+1} \qquad (n = 0, 1, \cdots).$$

For $p=-1$, (2.2) simplifies to $CP_n - mxP_{n-1} + yP_{n-m} = 0$, $n \geqq m \geqq 1$. Thus, if the roots of $Cz^m - mxz^{m-1} + y = 0$ are distinct, we obtain from (4.2)

$$\begin{aligned} &P_{nm}(m, x, y, -1, C) \\ &\quad = (-C)^{-n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} y^{n-k} (mx)^k P_{k(m-1)}(m, x, y, -1, C); \end{aligned} \tag{4.16}$$

using (4.13), we obtain from (4.16)

$$\begin{aligned} &(mx)^n P_{n(m-1)}(m, x, y, -1, C) \\ &\quad = \sum_{k=0}^{n} \binom{n}{k} y^{n-k} C^k P_{km}(m, x, y, -1, C). \end{aligned} \tag{4.17}$$

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University of Minnesota