A "METRIC" CHARACTERIZATION OF REFLEXIVITY[1]

JAMES P. WILLIAMS

Let $\{B, \|\ \|\}$ be a Banach space. The *dual norm* $\|\ \|^*$ is the "natural" norm on B^*:

$$\|f\|^* = \sup_{x \in B} \frac{|\langle x, f\rangle|}{\|x\|} \qquad (f \in B^*).$$

More generally, if $|x|$ is a norm on B which is equivalent to the given norm $\|x\|$, say

$$a\|x\| \leqq |x| \leqq b\|x\| \qquad (x \in B),$$

then

$$\frac{1}{b}\,\frac{|\langle x, f\rangle|}{\|x\|} \leqq \frac{|\langle x, f\rangle|}{|x|} \leqq \frac{1}{a}\,\frac{|\langle x, f\rangle|}{\|x\|} \qquad (x \in B, f \in B^*)$$

so that

$$|f|^* = \sup_{x \in B} \frac{|\langle x, f\rangle|}{|x|}$$

is finite for each $f \in B^*$. We call $|f|^*$ the *dual norm* of $|x|$.

It is clear from the above inequalities that if $|f|^*$ is a dual norm on B^*, then $|f|^*$ is equivalent to the natural norm $\|f\|^*$ on B^*. Thus, in general, each dual norm on B^* is an equivalent norm on B^*. The converse question is answered by the following theorem.

THEOREM 1. *Let $\{B, \|\ \|\}$ be given with dual space $\{B^*, \|\ \|^*\}$. Then every norm on B^* which is equivalent to $\|\ \|^*$ is a dual norm if and only if B is reflexive.*

The proof will make use of the following characterization of dual norms:

LEMMA. *Let p be a norm on B^* which is equivalent to the natural norm $\|\ \|^*$ on B^*. Then p is a dual norm if and only if the set*

$$U = \{f \in B^*: p(f) \leqq 1\}$$

is weak-star closed in B^*.

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PROOF. Observe first that if $p=|\ |^*$ is the dual of a norm $|\ |$ on B which is equivalent to the given norm $\|\ \|$ on B, then the norm $|x|$ of each $x \in B$ can be computed from p as follows (Proposition 5, Chapter IV, §5 of [1]):

$$|x| = \sup_{f \in B^*} \frac{|\langle x, f\rangle|}{|f|^*} = \sup_{f \in B^*} \frac{|\langle x, f\rangle|}{p(f)}.$$

(This also shows that p is the dual of at most one equivalent norm on B.)

Having made this observation, we now define $|x|$ in terms of p by the above formula. Our problem then becomes: Under what conditions on p do we have $p(f)=|f|^*$ for all $f \in B^*$?

Let $U^0=\{x \in B: |\langle x, f\rangle| \leqq 1 \text{ for all } f \in U\}$ be the polar of U in B. It follows from the definition of $|x|$ that $U^0=\{x \in B: |x| \leqq 1\}$. But then

$$U \subset U^{00} = \{x \in B: |x| \leqq 1\}^0 = \{f \in B^*: |f|^* \leqq 1\},$$

so that $p(f)=|f|^*$ $(f \in B^*)$ if and only if $U=U^{00}$. Since U is convex and $0 \in U$, the bipolar U^{00} of U can be identified as the weak-star closure of U and the lemma now follows at once.

COROLLARY. *If B is reflexive, then every norm p on B^* which is equivalent to $\|\ \|^*$ is a dual norm.*

PROOF. If p is equivalent to $\|\ \|^*$, then $U=\{f \in B^*: p(f) \leqq 1\}$ is convex, norm-closed and hence weakly closed. Since B is reflexive, the weak and weak-star topologies on B^* agree, and hence we can conclude from the Lemma that p is a dual norm on B^*.

REMARKS. 1. I am indebted to the referee for observing that the Lemma is not new. It follows directly from [2, Lemma 2 or Theorem 2].

2. Certainly one does not need the bipolar theorem to prove the sufficiency of Theorem 1. Indeed if $|\ |$ is an equivalent norm on the dual X^* of a reflexive space X, then $|\ |^*$ is an equivalent norm on $X^{**}=X$, and $|\ |$ is the norm dual to $|\ |^*$.

PROOF OF THE THEOREM. In view of the preceding corollary we need only prove the necessity part of the theorem. For this it suffices to show that if $X \in B^{**}$, then X is a weak-star continuous linear functional on B^*, or equivalently, that the null space N of X is weak-star closed in B^*. Clearly we may assume that X has norm 1.

By a result of Banach (Proposition 3, Chapter IV, §5 of [1]) N is weak-star closed in B^* if and only if its intersection V with the unit ball $\|f\|^* \leqq 1$ is weak-star closed in B^*. Therefore, setting

$$V_n = \{f \in B^*: |\langle X, f\rangle| \leqq 1/n \quad \text{and} \quad \|f\|^* \leqq 1\}$$

we have

$$V = \bigcap_{n=1}^{\infty} V_n,$$

and it suffices to show that each V_n ($n=1, 2, 3, \cdots$) is weak-star closed in B^*.

Now V_n is a convex, circled,[2] closed, bounded, neighborhood of 0 in B^*, and hence the gauge of V_n, namely the functional

$$p_n(f) = \inf \{\lambda > 0: f \in \lambda V_n\}$$

is an equivalent norm on B^*. The Lemma then implies that

$$\{f: p_n(f) \leqq 1\}$$

is weak-star closed in B^*. Since this latter set is V_n, we are done.

As the referee has pointed out, the proof of Theorem 1 contains several other characterizations of reflexivity. These are recorded in the following theorem:

THEOREM 2. *If X is a Banach space, then the following are equivalent:*

(1) *X is reflexive.*

(2) *Each equivalent norm on X^* is a dual norm.*

(3) *Each closed maximal vector subspace of X^* is weak-star closed.*

(4) *The norm closure of any bounded, convex, circled, neighborhood of 0 in X^* is also weak-star closed.*

(5) *Each bounded, convex, closed subset of B^* is weak-star compact.*

(6) *Each weakly closed subset of B^* is weak-star closed.*

PROOF. The proof of Theorem 1 shows that each of (2)–(6) is sufficient for reflexivity. Thus condition (3) implies that N is weak-star closed, and each of (2), (4), (5), (6) implies that V_n is weak-star closed. Conversely, we proved that (2) is necessary, and the necessity of (4), (5) and (6) follows from the identity of weak and weak-star topologies in a reflexive space. Condition (3) is also necessary since a closed maximal vector subspace of X^* is the null space of a bounded linear functional on X^*.

REFERENCES

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INSTITUTE OF SCIENCE AND TECHNOLOGY, THE UNIVERSITY OF MICHIGAN AND INDIANA UNIVERSITY

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[2] A set V is *circled* if $\alpha V \subset V$ for each scalar α of modulus one.