EXTREME EIGENVECTORS OF A NORMAL OPERATOR

S. J. BERNAU

In a recent paper [3] MacCluer proves that if A is a bounded normal operator on a complex Hilbert space $\boldsymbol{H}$ then the extreme points of the numerical range of A are eigenvalues of A. MacCluer's proof uses the spectral theorem in an essential fashion. Our purpose is to point out that a slightly stronger result is elementary, independent of the spectral theorem and true for unbounded normal operators. We show that if $\|x\|=1$ and $\lambda=(Ax, x)$ is an extreme point of the numerical range then $Ax=\lambda x$.

THEOREM. *Let A be a not necessarily bounded, normal operator on a complex Hilbert space $\boldsymbol{H}$, and let $W(A)=\{(Ax, x): x\in\mathfrak{D}(A), \|x\|=1\}$. If λ is an extreme point of $W(A)$, $x\in\mathfrak{D}(A)$, $\|x\|=1$ and $(Ax, x)=\lambda$, then $Ax=\lambda x$.*

(Recall that a closed operator A is normal if its domain $\mathfrak{D}(A)$ is dense in $\boldsymbol{H}$ and $AA^*=A^*A$.)

PROOF. Because $W(e^{i\theta}(A-\lambda I))=e^{i\theta}(W(A)-\lambda)$ we may assume that $\lambda=0$ and $W(A)$ is contained in the closed right half plane $\{z\colon \operatorname{Re} z\geqq 0\}$. Clearly $A+A^*$ is symmetric and $W(A+A^*)$ is a set of nonnegative real numbers; thus, by the generalized Schwarz inequality [4, §104].

$$(1)\quad |((A+A^*)x, y)|^2 \leqq ((A+A^*)x, x)((A+A^*)y, y)(x, y\in\mathfrak{D}(A)).$$

Suppose that $x\in\mathfrak{D}(A)$, $\|x\|=1$ and $(Ax, x)=0$. It follows that $(A^*x, x)=0$ and hence, by (1), $((A+A^*)x, y)=0$ $(y\in\mathfrak{D}(A))$. Because $\mathfrak{D}(A)$ is dense, $(A+A^*)x=0$. Let M be the (nontrivial) null space of $A+A^*$. For y in M

$$(Ay, y) = (y, A^*y) = -(y, Ay) = -(Ay, y)^*,$$

so that $\operatorname{Re}(Ay, y)=0$ $(y\in M)$. Because 0 is an extreme point of $W(A)$ it follows that $\operatorname{Im}(Ay, y)$ is either nonnegative, or nonpositive for all y in M. Suppose that $\operatorname{Im}(Ay, y)\geqq 0$ $(y\in M)$. Then, for $S=-iA$ we have $(Sy, y)\geqq 0$ $(y\in M)$. Because $x\in\mathfrak{D}(A)$, $\|x\|=1$ and $(Ax, x)=0$ we have, $(Sx, x)=0$ and, by the generalized Schwarz inequality again,

$$(2)\qquad (Sx, y) = 0 \qquad (y\in M).$$

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If A is bounded we see at once that M is closed and invariant under A. Hence $Sx=-iAx\in M$ and $Ax=0$.

If A is not bounded we proceed as follows. Because A is normal $(A+A^*)^{**}$, the closure of $A+A^*$, is self adjoint. It follows that

$$M^{\perp}=\mathrm{cl}\,\mathcal{R}((A+A^*)^{**})=\mathrm{cl}\,\mathcal{R}(A+A^*).$$

Now let $z\in\mathfrak{D}(A)$, by [2, Lemma XII.7.1], there is a sequence (z_n) in $\mathfrak{D}(A^*A)$ such that $z_n\to z$ and $Az_n\to Az$. Because A is normal $A^*z_n\to A^*z$ and hence

$$\begin{aligned}(Sx,(A+A^*)z)&=\lim(-iAx,(A+A^*)z_n)\\&=\lim(-ix,A(A+A^*)z_n)\\&=\lim(-ix,(A+A^*)Az_n)\\&=\lim(-i(A+A^*)x,Az_n)\\&=0.\end{aligned}$$

Thus $Sx\in M^{\perp\perp}=\overline{M}$ and, by (2), $Ax=iSx=0$. This proves our theorem.

In conclusion we remark that the result that, for normal A, $(A+A^*)^{**}$ is self adjoint is given as an exercise in [2, XII.9.11]. An elementary proof can be based on the polar decomposition for a normal operator [2, XII.9.10]. This in turn depends on the existence of a square root for an unbounded positive self adjoint operator. An elementary proof of this and of the polar decomposition are given in [1].

ADDED IN PROOF. The bounded case of our theorem is obtained, with essentially the same proof, by S. Hildebrandt, *Über den numerischen Wertebereich eines Operators*, Math. Ann. **163** (1966), 230–247 (Lemma 11) and is originally due to W. F. Donoghue, *On the numerical range of a bounded operator*, Michigan Math. J. **4** (1957), 261–263. Neither of these references was known to the author when this paper was written.

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UNIVERSITY OF CANTERBURY, CHRISTCHURCH, NEW ZEALAND