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REPRESENTATION OF 0 AS $\sum_{k=-N}^{N} \epsilon_k k$

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Abstract. If ϵ_k are independent identically distributed random variables with values 0 and 1, each with probability $\frac{1}{2}$ then

$$P\left(\sum_{k=-N}^{+N} \epsilon_k k = 0\right) \sim \left(\frac{3}{\pi}\right)^{1/2} N^{-3/2}.$$

1. **Introduction.** Recently P. Erdös asked the following question (oral communication[1]). If $A(N)$ denotes the number of representations of 0 in the form $\sum_{k=-N}^{N} \epsilon_k k$, where $\epsilon_k = 0$ or 1 for $-N \leqq k \leqq N$ then determine the asymptotic behavior of $A(N)$. We shall prove that

$$A(N) \sim \left(\frac{3}{\pi}\right)^{1/2} 2^{2N+1} N^{-3/2}. \tag{1}$$

Another way of formulating this result is the following. Let ϵ_k be independent random variables identically distributed with values 0 and 1, each with probability $\frac{1}{2}$. Then

$$P\left(\sum_{k=-N}^{N} \epsilon_k k = 0\right) \sim \left(\frac{3}{\pi}\right)^{1/2} N^{-3/2}. \tag{2}$$

The referee has pointed out that (2) can be expected from the Lindeberg theorem as follows. If N is large $\sum_{k=-N}^{N} \epsilon_k k$ is approximately normally distributed with mean zero and variance $N(N+1)(2N+1)/12$. The right hand side of (2) is the probability density at the origin of

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[1] See also [1].

the normal distribution with the same mean and variance.

2. **Proof of** (1). The number $A(N)$ is the constant term in the expansion of $\prod_{k=-N}^{N}(1+x^k)$. Hence

$$(3)\quad A(N)=\frac{1}{2\pi i}\int_C\prod_{k=-N}^{N}(1+z^k)\frac{dz}{z}=\frac{2^{2N+2}}{\pi}\int_0^{\pi/2}\prod_{k=1}^{N}\cos^2 kx\,dx$$

(here C is the unit circle in the complex plane).

Now first we estimate the integrand for values of x not near to the origin.

$$\begin{aligned}\prod_{k=1}^{N}\cos^2 kx&=\prod_{k=1}^{N}(1-\sin^2 kx)<\exp\left[-\sum_{k=1}^{N}\sin^2 kx\right]\\ (4)\qquad &=\exp\left[-\frac{N}{2}+\frac{\sin Nx\cos(N+1)x}{2\sin x}\right]\\ &=O(e^{-N/4})\qquad\text{for } \frac{\pi}{2N}\leqq x\leqq\frac{\pi}{2}\cdot\end{aligned}$$

Next we remark that $\cos^2 x<e^{-x^2}$ for $0\leqq x\leqq\pi/2$. Therefore

$$\begin{aligned}\int_0^{\pi/2N}\prod_{k=1}^{N}\cos^2 kx\,dx&<\int_0^{\pi/2N}\exp\left[-\sum_{k=1}^{N}k^2x^2\right]dx\\ (5)\qquad &=\int_0^{\pi/2N}\exp\left[-\frac{N(N+1)(2N+1)}{6}x^2\right]dx\\ &\sim\left(\frac{3}{N^3}\right)^{1/2}\int_0^{\infty}e^{-t^2}dt=\frac{1}{2}(3\pi)^{1/2}N^{-3/2}.\end{aligned}$$

Now for $0\leqq x<N^{-4/3}$ we have

$$\begin{aligned}\prod_{k=1}^{N}\cos^2 kx&=\prod_{k=1}^{N}e^{-k^2x^2}\prod_{k=1}^{N}\{1+O(k^4x^4)\}\\ &=\prod_{k=1}^{N}e^{-k^2x^2}\exp\{O(N^5N^{-16/3})\}\\ &=\exp\left\{-\sum_{k=1}^{N}k^2x^2+O(N^{-1/3})\right\}\cdot\end{aligned}$$

Using the fact that

$$\int_0^{N^{-4/3}}\exp\left(-\sum_{k=1}^{N}k^2x^2\right)dx\sim\frac{1}{2}(3\pi)^{1/2}N^{-3/2}$$

and the result (5) we find

$$\int_0^{\pi/2N} \prod_{k=1}^{N} \cos^2 kx dx \sim \frac{1}{2}(3\pi)^{1/2} N^{-3/2}. \tag{6}$$

Now from (6), (4) and (3) we find

$$A(N) \sim \left(\frac{3}{\pi}\right)^{1/2} 2^{2N+1} N^{-3/2}.$$

This completes the proof.

3. **Acknowledgment.** The author is indebted to B. F. Logan for a valuable suggestion which led to (4).

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