LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS WITH PERIODIC SOLUTIONS[1]

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1. **Introduction.** Epstein [1] has shown that the system of differential equations

$$x' = A(t)x, \quad t \in (-\infty, \infty), \quad \left(' = \frac{d}{dt}\right), \tag{1}$$

where x is a column vector and A is an $n \times n$ matrix, has solutions which are all periodic with period ω when the entries in A are odd continuous functions of t which are periodic with period ω. The present paper gives generalizations of this result in which the condition of oddness on A is relaxed considerably. Our results apply to matrices A which, in addition to periodicity, have the property that there can be associated with (1) an equation

$$y' = B(t)y, \tag{2}$$

not necessarily different from (1), such that there are two changes of variables satisfying certain conditions which transform (1) into (2) or (2) into (1).

2. **Results.**

THEOREM 1. *Assume that the matrix A is continuous or piecewise continuous, that*

$$A(t + \omega) = A(t) \tag{3}$$

and that the following conditions are satisfied:

(i) *There exists a matrix B which is given a.e. on an interval I by*

$$\Phi_i' \Phi_i^{-1} + f_i' \Phi_i A(f_i) \Phi_i^{-1} = B, \qquad i = 1, 2, \tag{4}$$

where Φ_1 and Φ_2 are nonsingular matrices, f_1 and f_2 are real-valued functions, the entries of Φ_i and the functions f_i being absolutely continuous on I and the primes denote derivatives.

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(ii) *There exist points* t_1 *and* t_2 *in* I *such that*

$$f_i(t_i) = f_j(t_i) + m_i\omega, \qquad i = 1, 2 \quad \textit{and} \quad i \neq j \tag{5}$$

where $m_1 = 0$ *and* $m_i = \pm 1$, *and*

$$\Phi_1(t_i) = c\Phi_2(t_i), \qquad i = 1, 2, \tag{6}$$

where c *is a scalar constant.*

Then every solution of (1) *is periodic with period* ω.

The following corollary is obtained by taking $\Phi_1 = \Phi_2 = E$ (the unit matrix) in the above theorem.

COROLLARY 1. *If* A *continuous or piecewise continuous and there exist absolutely continuous functions* f_1 *and* f_2 *on some interval* I *such that*

$$f_1' A(f_1) = f_2' A(f_2), \qquad \textit{a.e. in } I, \tag{4)$'$}$$

and if there exist points t_1 *and* t_2 *in* I *at which* (5) *holds then every solution of* (1) *is periodic with period* ω.

PROOF OF THEOREM 1. Let X be a fundamental solution matrix of (1). It is a well known result of Floquet Theory that (3) implies

$$X(t + m\omega) = X(t)V^m, \quad \text{for all } t, \tag{7}$$

where m is any integer and V is a constant nonsingular matrix. Let $Y_i = \Phi_i X(f_i)$, $i = 1, 2$; then

$$\begin{aligned} Y_i' &= \Phi_i' X(f_i) + f_i' \Phi_i \frac{d}{df_i} X(f_i), \quad \text{a.e. in } I, \\ &= \Phi_i' X(f_i) + f_i' \Phi_i A(f_i) X(f_i), \quad \text{from (1)}, \\ &= \Phi_i' \Phi_i^{-1} Y_i + f_i' \Phi_i A(f_i) \Phi_i^{-1} Y_i \\ &= B Y_i, \quad \text{by (4)}. \end{aligned}$$

Y_i is nonsingular since Φ_i and X are nonsingular so that Y_1 and Y_2 are fundamental solution matrices of (2). Consider

$$\begin{aligned} Y_i(t_i) &= \Phi_i(t_i) X(f_i(t_i)) \\ &= \Phi_i(t_i) X(f_j(t_i) + m_i\omega), \quad \text{by (5)}, \\ &= \Phi_i(t_i) X(f_j(t_i)) V^{m_i}, \quad \text{by (7)}, \\ &= \Phi_i(t_i) \Phi_j^{-1}(t_i) Y_j(t_i) V^{m_i}, \end{aligned}$$

so that $Y_1(t_1) = cY_2(t_1)V^{m_1} = cY_2(t_1)$ and $Y_2(t_2) = (1/c)Y_1(t_2)V^{m_2}$, by (6). We note that the piecewise continuity of A and absolute continuity of Φ_i and f_i are sufficient conditions for the uniqueness of solutions of (2) and hence $Y_1 = cY_2$ and $Y_2 = (1/c)\, Y_1 V^{m_2}$ everywhere in I since both sides of either of these equations are solutions of (2) with the same initial conditions at a point of I. Substitution of one of these equations into the other gives $Y_2 = Y_2 V^{m_2}$ and $E = V^{m_2}$ since Y_2 is nonsingular. Thus $V = E$ since $m_2 = \pm 1$ and (7) implies that $X(t+\omega) = X(t)$ for all t. Q.E.D.

The various steps of the proof of Theorem 1 also hold if m_1 and m_2 are possibly integers other than 0 and ± 1 and $X(t+(m_1+m_2)\omega) = X(t)$. However it may be assumed without loss of generality that $m_1 = 0$ and $m_2 \geqq 1$; the Bolzano-Weierstrass intermediate value theorem for continuous functions with (5) applied to $f = f_2 - f_1$ shows the existence of a point t_2' such that $f_2(t_2') = f_1(t_2') + \omega$ and this is a special case of Theorem 1 so that $X(t+\omega) = X(t)$.

THEOREM 2. *Assume that there exists a continuous or piecewise continuous matrix B such that the following conditions are satisfied:*

(i) *A is given a.e. on two intervals J_1 and J_2 by*

$$\Psi_i' \Psi_i^{-1} + g_i' \Psi_i B(g_i) \Psi_i^{-1} = A, \qquad i = 1, 2, \tag{8}$$

where Ψ_1 and Ψ_2 are nonsingular matrices, g_1 and g_2 are real-valued functions on J_1 and J_2 respectively, the entries of Ψ_i and the functions g_i being absolutely continuous on J_i.

(ii) *There is a point τ_i in J_i such that $\tau_i + n_i\omega$ is in J_j,*

$$g_i(\tau_i) = g_j(\tau_i + n_i\omega), \qquad i = 1, 2 \quad \text{and} \quad i \neq j, \tag{9}$$

where $n_1 = 0$ and $n_2 = \pm 1$;

$$\Psi_1(\tau_i + n_i\omega) = k\Psi_2(\tau_i), \qquad i = 1, 2, \tag{10}$$

where k is a scalar constant.

Then if A is periodic with period ω every solution of (1) *is periodic with period ω.*

The case $\Psi_1 = E$ and $\Psi_2 = E$ may be stated as the following corollary.

COROLLARY 2. *If* (3) *holds and there is a piecewise continuous matrix B and absolutely continuous functions g_1 and g_2 on intervals J_1 and J_2 such that*

$$g_i' B(g_i) = A, \quad \text{a.e. in } J_i, \qquad i = 1, 2 \tag{8$'$}$$

and if (9) *holds at points τ_i in J_i then every solution of* (1) *is periodic with period ω.*

PROOF OF THEOREM 2. Let Y be a fundamental solution matrix of (2) and let $X_i = \Psi_i Y(g_i)$, then as in the proof of Theorem 1, one finds that X_1 and X_2 are fundamental solution matrices of (1) in J_1 and J_2 respectively. Consider

$$\begin{aligned} X_i(\tau_i) &= \Psi_i(\tau_i)\, Y(g_i(\tau_i)) \\ &= \Psi_i(\tau_i)\, Y(g_j(\tau_i + n_i\omega)), \quad \text{by (9)}, \\ &= \Psi_i(\tau_i)\Psi_j^{-1}(\tau_i + n_i\omega) X_j(\tau_i + n_i\omega) \\ &= \Psi_i(\tau_i)\Psi_j^{-1}(\tau_i + n_i\omega) X_j(\tau_i) V^{n_i}, \quad \text{by (7)}, \end{aligned}$$

so that $X_1(\tau_1) = kX_2(\tau_1)$ and $X_2(\tau_2) = (1/k)X_1(\tau_2)\, V^{n_2}$, by (10). Hence $X_1 = kX_2$ and $X_2 = (1/k)X_1 V^{n_2}$ for all t and $X_2 = X_2 V^{n_2}$ so that $V^{n_2} = E$ and $V = E$ since $n_2 = \pm 1$. Therefore (7) implies that every solution of (1) is periodic with period ω. Q.E.D.

3. **Remarks.** (i) In the case that either f_1 and f_2 or g_1 and g_2 are monotonic Theorems 1 and 2 are statements of the same results; in this case also Corollaries 1 and 2 are equivalent. For example, if f_1 and f_2 are monotonic we will show that the conditions (4), (5) and (6) of Theorem 1 may be written in the form of (8), (9) and (10), respectively, of Theorem 2.

Let $J_i = f_i(I)$ and define $g_i = f_i^{-1}$ (the inverse function of f_i) and $\Psi_i = \Phi_i^{-1}(g_i)$ (Φ_i^{-1} is the multiplicative inverse of Φ_i). Equation (4) may be written

$$-\Phi_i^{-1}\Phi_i' + \Phi_i^{-1}B\Phi_i = f_i' A(f_i).$$

Now

$$A(f_i(g_i)) = A; \frac{d}{dg_i} f_i(g_i) = \frac{1}{g_i'}$$

since $1 = [f_i(g_i)]' = g_i'(d/dg_i)f_i(g_i)$; also

$$\Psi_i' = -g_i'\Psi_i\left[\frac{d}{dg_i}\ \Phi_i(g_i)\right]\Psi_i$$

since $(\Phi_i^{-1})' = -\Phi_i^{-1}\Phi_i'\Phi_i^{-1}$. Hence (4) may be expressed as

$$\Psi_i'\Psi_i^{-1} + g_i'\Psi_i B(g_i)\Psi_i^{-1} = A$$

which is (8). If we take $\tau_i = f_i(t_i)$ then (5) and (6) may be written

$$g_i(\tau_i) = g_j(\tau_i - m_i\omega), \qquad i = 1, 2 \text{ and } i \neq j$$

and

$$\Psi_1(\tau_i - m_i\omega) = \frac{1}{c}\Psi_2(\tau_i)$$

which are (9) and (10) respectively with $n_i = -m_i$ and $k = 1/c$.

(ii) The result of Epstein mentioned in the introduction is given, for example, by Corollary 1 with $f_1(t)=t$ and $f_2(t)=-t$. In this case we may take $t_1=0$ and $t_2=-\omega/2$ so that $m_1=0$ and $m_2=+1$.

4. **Examples.** We conclude by constructing examples of matrices A on a typical period $[0, \omega]$ for which (1) has periodic solutions and which illustrate a few of the different types of systems to which the theorems apply.

(i) Let I be the interval $[0, a]$, $0<a<\omega$, and define A on the period $[0, \omega]$ as follows: on $[a, \omega]$ let A be any continuous square matrix and on $[0, a)$ let

$$A = \Phi_1^{-1}B\Phi_1 - \Phi_1^{-1}\Phi_1',$$

where

$$B = f'\Phi_2 A(f)\Phi_2^{-1} + \Phi_2'\Phi_2^{-1},$$

Φ_1 and Φ_2 are nonsingular matrices continuously differentiable on $[0, a]$ with $\Phi_1=\Phi_2$ at 0 and a and f is a continuously differentiable function on $[0, a]$ onto $[a, \omega]$ with $f(0)=\omega$ and $f(a)=a$. The matrix A thus defined is piecewise continuous on $(-\infty, \infty)$ and conditions (4), (5) and (6) of Theorem 1 hold if we take $f_1(t)=t$, $f_2(t)=f(t)$, $t_1=a$, $t_2=0$ so that $c=1$, $m_1=0$ and $m_2=1$ and all solutions of (1) are periodic with period ω.

Note that in this example equations (1) and (2) are the same on the interval I in the case $\Phi_1=E$.

(ii) Let g be any decreasing continuously differentiable function on $[0, a]$, $0<a<\omega$, with $g(0)=\omega$ and $g(a)=a$. Let A be any continuous square matrix on $[a, \omega]$ and define A on $[0, a)$ by

$$A(t) = g'(t)A(g(t)), \qquad 0 \leqq t < a.$$

This equation may also be written

$$A(t) = (g^{-1}(t))'A(g^{-1}(t)), \qquad a < t \leqq \omega.$$

where g^{-1} is the inverse function of g. Hence if $f(t)=g(t)$, $0\leqq t\leqq a$ and $f(t)=g^{-1}(t)$, $a\leqq t\leqq\omega$, then

$$A(t) = f'(t)A(f(t)), \qquad 0 \leqq t \leqq \omega, \quad t \neq a.$$

To apply Corollary 1 to A we may choose $I=[0, a]$, $[a, \omega]$ or $[0, \omega]$ with $f_1(t)=t$, $f_2(t)=f(t)$, $t_1=a$, $t_2=0$ or ω, $m_1=0$ and $m_2=\pm 1$, the choice of t_2 and the value of m_2 depending on the interval I chosen. Thus every solution of (1) is periodic with period ω.

(iii) Let B be any continuous square matrix on $[a, b]$; let g be any absolutely continuous function on $[0, \omega]$, $a < g < b$ and $g(0) = g(\omega)$. Define A a.e. on $[0, \omega]$ by

$$A = g'B(g).$$

Let $J_1 = J_2 = [0, \omega]$, $g_1 = g_2 = g$, $\tau_1 = c$ and $\tau_2 = 0$, where c is any point of $[0, \omega]$; we may take $n_1 = 0$ and $n_2 = 1$ and use Corollary 2 to show that in this example also every solution of (1) is periodic with period ω.

Reference

1. I. J. Epstein, *Periodic solutions of systems of differential equations*, Proc. Amer. Math. Soc. **13** (1962), 690–694.

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