A COMBINATORIAL PROBLEM IN THE k-ADIC NUMBER SYSTEM

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1. **Introduction.** Let N denote the set of all nonnegative integers. The elements in N are represented in the k-adic number system by strings of integers as $a_1a_2 \cdots a_p$, $0 \leqq a_\nu \leqq k-1$. Define a multivalued function on N by

$$\Gamma(a_1a_2 \cdots a_p) = \{a_1 \cdots (a_\nu - 1) \cdots a_p;\ 1 \leqq \nu \leqq p,\ a_\nu \geqq 1\}$$

and $\Gamma(0) = \emptyset$, the null set. Put $\alpha_k(a_1a_2 \cdots a_p) = \sum a_\nu$, $\nu = 1, 2, \cdots, p$ and $\alpha_k(S) = \sum \alpha_k(n)$, $n \in S$ if $S \subset N$.

S is said to be *closed* if $S \subset N$ and $\Gamma S \subset S$. $S_n = \{0, 1, \cdots, n-1\}$ is closed. The problem is to determine the maximum of $\alpha_k(S)$ when S ranges over all closed S with $|S| = n$, i.e. with n elements. Our main result (Theorem 1) is that the maximum is $\alpha_k(S_n)$.

If we put $B_k(n) = \alpha_k(S_n)$, we get as a corollary

$$B_k(m_1 + m_2 + \cdots + m_k) \geqq \sum_{\nu=1}^{k} B_k(m_\nu) + \sum_{\nu=2}^{k} (\nu - 1)m_\nu,$$
$$m_1 \geqq m_2 \geqq \cdots \geqq m_k \geqq 0.$$

It is interesting that Theorem 1 can be derived from this inequality. We have no independent proof of it, except for $k=2$.

The asymptotic properties of the function $A_k(n) = B_k(n+1)$ were studied in [1] by R. Bellman and H. N. Shapiro. $A_2(n)$ appeared in connection with determinants in [2]. A result in that paper will be extended in our Theorem 2. We also note that there is some connection with the "detecting sets" studied in [3]. In fact, it was an attempt to extend the results in [3] which gave rise to the present problem.

2. **Main results.** In this section we shall derive the following theorem:

THEOREM 1. *If S is closed and $|S| = n$, then $\alpha_k(S) \leqq \alpha_k(S_n)$.*

To simplify notations we shall omit the index "k" in the proofs.

Putting 0's in front of a string does not alter the integer represented by the string. Hence we can assume that all integers in S are represented by strings of the same length $p = p(S)$.

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Given $S \subset N$, we shall define a set $S^c \subset N$, called the *compression* of S. Let S_ν denote the set of all integers $n \in S$ for which $\alpha(n) = \nu$. Let S_ν^c denote the set of the $|S_\nu|$ smallest nonnegative integers n for which $\alpha(n) = \nu$. Then define S^c as the union of the sets S_ν^c, $\nu = 0, 1, 2, \cdots$. We note that

$$|S^c| = |S|, \tag{2.1}$$

$$\alpha(S^c) = \alpha(S). \tag{2.2}$$

We shall prove a lemma:

LEMMA 1. *If $p(S) = 2$ and S is closed, then S^c is closed.*

PROOF. It is instructive to imagine the integers $a_1a_2 \in S$ as points with coordinates (a_1, a_2) in a 2-dimensional coordinate-system.

If $a_1 \neq 0$ and $a_2 \neq 0$ for every $a_1a_2 \in S_\nu$ (or S_ν^c), then

$$|\Gamma S_\nu| \geqq |S_\nu| + 1 \quad \text{and} \quad |\Gamma S_\nu^c| = |S_\nu^c| + 1. \tag{2.3}$$

This holds surely when $\nu \geqq k$.

If there is one and only one integer $a_1a_2 \in S_\nu$ (or S_ν^c) for which a_1 or $a_2 = 0$, then we find

$$|\Gamma S_\nu| \geqq |S_\nu| \quad \text{and} \quad |\Gamma S_\nu^c| = |S_\nu^c|. \tag{2.4}$$

From (2.3) and (2.4) we get in both cases

$$|\Gamma S_\nu^c| \leqq |\Gamma S_\nu|. \tag{2.5}$$

If there are two integers a_1a_2 for which a_1 or $a_2 = 0$ then $S_\nu = S_\nu^c$ and (2.5) holds with equality.

S is closed if and only if $\Gamma S_\nu \subset S_{\nu-1}$ for $\nu = 1, 2, \cdots$. Then we find by (2.1) and (2.5)

$$|\Gamma S_\nu^c| \leqq |S_{\nu-1}^c|, \qquad \nu = 1, 2, \cdots.$$

From this inequality it follows $\Gamma S_\nu^c \subset S_{\nu-1}$ for $\nu = 1, 2, \cdots$. Hence S^c is closed and the lemma is proved.

We shall prove a second lemma

LEMMA 2. *Assume $p = p(S) \geqq 3$ for $S \subset N$, and that $b_1b_2 \cdots b_p \in S$, $a_i = b_i$ and $a_1 \cdots a_{i-1}a_{i+1} \cdots a_p < b_1 \cdots b_{i-1}b_{i+1} \cdots b_p$ implies $a_1 \cdots a_p \in S$ for $i = 1, 2, \cdots, p$. Then $b_1b_2 \cdots b_p \in S$, $a_1a_2 \cdots a_p < b_1b_2 \cdots b_p$ and $a_1 + \cdots + a_p \leqq b_1 + \cdots + b_p$ implies $a_1a_2 \cdots a_p \in S$.*

PROOF. We can assume $a_\nu \neq b_\nu$, $1 \leqq \nu \leqq p$. Then $a_1 < b_1$, since $a_1a_2 \cdots a_p < b_1b_2 \cdots b_p$. If there is $s \neq 1$ such that $a_s < b_s$, we get

$$b_1 \cdots b_s \cdots b_p > b_1 \cdots a_s \cdots a_p > a_1 \cdots a_s \cdots a_p.$$

From these inequalities we find $a_1a_2 \cdots a_p \in S$ if $b_1b_2 \cdots b_p \in S$.

Next we assume $a_\nu > b_\nu$ for $\nu > 1$. Since $a_1 + \cdots + a_p \leqq b_1 + \cdots + b_p$, we get $b_1 - a_1 \geqq (a_2 - b_2) + \cdots + (a_p - b_p) \geqq p - 1 \geqq 2$. Hence

$$b_1b_2 \cdots b_p > (b_1 - 1)a_2b_3 \cdots b_p > (b_1 - 2)a_2 \cdots a_p \geqq a_1a_2 \cdots a_p.$$

Then from $b_1b_2 \cdots b_p \in S$ we conclude $a_1a_2 \cdots a_p \in S$.

Proof of Theorem 1. The proof is by induction over $p = p(S)$. If $p = 1$, $S = S_n$ and the theorem is true. Next we assume $p = 2$. The compressed set S^c is formed from S. If $S^c \neq S_n$ let a_1a_2 be the smallest nonnegative integer not in S^c and let b_1b_2 be the largest integer in S^c. Then we find $a_1a_2 < b_1b_2$, $a_1 < b_1$, $a_2 > b_2$, for S^c is closed by Lemma 1. We get even

$$a_1 + a_2 > b_1 + b_2. \tag{2.6}$$

For if $a_1 + a_2 \leqq b_1 + b_2$, we can put $c = a_1 + a_2 - b_2$. Then $a_1 < c \leqq b_1$ and $cb_2 \in S^c$ for S^c is closed. Hence $a_1a_2 \in S^c$, since $a_1 + a_2 = c + b_2$ and S^c is compressed. But $a_1a_2 \notin S^c$, and (2.6) follows by the contradiction.

If b_1b_2 is deleted from S^c and a_1a_2 is adjoined to it, we get a new closed and compressed set T. We find by (2.1) and (2.2)

$$|T| = |S|, \qquad \alpha(T) > \alpha(S). \tag{2.7}$$

If $T \neq S_n$ we can find new integers a_1a_2 and b_1b_2. After a finite number of steps we get S_n, for the sum of all integers in the set is decreased at each step. By (2.7) the theorem holds for $p = 2$.

Now we assume that T is a closed set with $p = p(T) \geqq 3$. For a_1 fixed we shall consider the set $T(a_1) = \{a_2a_3 \cdots a_p;\, a_1a_2 \cdots a_p \in T\}$. $T(a_1)$ is closed and $p(T(a_1)) = p - 1$. By assumption the theorem holds for $T(a_1)$. Replace $T(a_1)$ by a set S_n, $n = |T(a_1)|$, restore the digit a_1 and take union when $a_1 = 0, 1, \cdots, k-1$. We get T_1 with $\alpha(T_1) \geqq \alpha(T)$. Note that $|T(\nu - 1)| \geqq |T(\nu)|$, since T is closed. It follows that T_1 is closed. Define $T_1(a_2) = \{a_1a_3 \cdots a_p;\, a_1a_2 \cdots a_p \in T_1\}$. $T_1(a_2)$ is closed. Replace it by a set of type S_n, restore the digit a_2 and take union when $a_2 = 0, 1, \cdots, k-1$. T_2 is closed and $\alpha(T_2) \geqq \alpha(T_1)$. Continue with the digits $a_3, \cdots, a_p, a_1, a_2, \cdots$. We get a sequence of closed sets: $T, T_1, T_2, \cdots$, for which

$$\alpha(T_{m+1}) \geqq \alpha(T_m), \qquad |T_m| = |T|. \tag{2.8}$$

If $T_{m+1} \neq T_m$, then the sum of all integers in T_{m+1} is smaller than the sum of all integers in T_m. Hence there is an index q such that

$$T_q = T_{q+1} = \cdots = T_{q+p}.$$

Then we find that T_q meets the requirements on S in Lemma 2. If $T_q \neq S_n$, $n = |T|$, we can find a minimal $a_1 a_2 \cdots a_p \notin T_q$ and a maximal $b_1 b_2 \cdots b_p \in T_q$ such that $a_1 \cdots a_p < b_1 \cdots b_p$ and, by Lemma 2,

$$a_1 + a_2 + \cdots + a_p > b_1 + b_2 + \cdots + b_p.$$

We delete $b_1 b_2 \cdots b_p$ from T_q and adjoin $a_1 a_2 \cdots a_p$ to the set. Then we get a closed set U for which $\alpha(U) > \alpha(T_q)$. U fulfills the requirements on S in Lemma 2. If $U \neq S_n$ we proceed to a new closed set with larger α-value. After a finite number of steps we get S_n. Hence $\alpha(T) \leqq \alpha(S_n)$ and the theorem follows by induction over p.

It is interesting to know that Lemma 1 is not valid for $p(S) > 2$. This is seen by the example:

$$S = \{000, 001, 010, 100, 002, 011, 020, 110, 012, 021, 120\},$$
$$S^c = \{000, 001, 010, 100, 002, 011, 020, 101, 012, 021, 111\}.$$

S is closed, but S^c is not closed since $110 \in \Gamma S^c$ and $110 \notin S^c$.

COROLLARY.

$$B_k(m_1 + \cdots + m_k) \geqq \sum_{\nu=1}^{k} B_k(m_\nu) + \sum_{\nu=2}^{k} (\nu - 1)m_\nu,$$

$$m_1 \geqq m_2 \geqq \cdots \geqq m_k \geqq 0.$$

$$B_k(mn) \geqq mB_k(n) + nB_k(m), \qquad m, n \geqq 1.$$

PROOF. Determine p such that $m_1 \leqq k^p$ and consider the set

$$S = \bigcup_{\nu=1}^{k} \{a_1 a_2 \cdots a_p(\nu - 1); a_1 a_2 \cdots a_p \in S_{m_\nu}\}.$$

S is closed and $|S| = m_1 + \cdots + m_k$. The first inequality follows if we determine $\alpha(S)$ and apply Theorem 1.

The second inequality follows if we determine p and q such that $m \leqq k^p$ and $n \leqq k^q$ and consider the set

$$T = \{a_1 \cdots a_p b_1 \cdots b_q; a_1 \cdots a_p \in S_m, b_1 \cdots b_q \in S_n\}.$$

T is closed, $|T| = mn$, $\alpha(T) = m\alpha(S_n) + n\alpha(S_m)$ and $\alpha(T) \leqq \alpha(S_{mn})$.

3. **Application to determinants.** We assume here that $k = 2$. There is a one-one mapping from nonnegative integers to sets of nonnegative integers:

$$n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_t} \to N = \{n_1, n_2, \cdots, n_t\},$$
$$n_1 > n_2 > \cdots > n_t \geqq 0, \tag{3.1}$$
$$0 \to \varnothing.$$

The set-theoretic counterpart to closed set of integers is closed family of sets: $\mathfrak{F}$ is a closed family of sets if $N \in \mathfrak{F}$, $M \subset N$ implies $M \in \mathfrak{F}$.

Put $\alpha(N) = |N|$ and $\alpha(\mathfrak{F}) = \sum \alpha(N)$, $N \in \mathfrak{F}$. For functions f defined on a closed family $\mathfrak{F}$, we put

$$\hat{f}(N) = \sum_{M \subset N} (-1)^{|M|} f(M), \tag{3.2}$$

where the sum is taken over all subsets to N. It is easy to verify $(\hat{f})\hat{\ } = f$. The proof of the following lemma can also be omitted (cf. [3, p. 481]).

LEMMA 3. *If f is defined on a closed family $\mathfrak{F}$, and M, $N \in \mathfrak{F}$, $M \not\subset N$,*

$$\sum_{S \subset M} (-1)^{|S|} f(S \cap N) = 0.$$

We shall prove the theorem on determinants:

THEOREM 2. *Let $N_1, N_2, \cdots, N_n$ be an enumeration of all sets in a closed family for which $N_i \subset N_j$ only if $i \leqq j$. Then*

$$|\hat{f}(N_i \cap N_j)|_{i,j=1}^{n} = \prod_{i=1}^{n} (-1)^{|N_i|} f(N_i).$$

PROOF. Multiply the last row in the determinant by $(-1)^{|N_n|}$. If $N_i \subset N_n$ we multiply the ith row by $(-1)^{|N_i|}$ and add to the last row. In the last row of the new determinant are all entries 0, except the last one which is $(\hat{f})\hat{\ }(N_n) = f(N_n)$. The value of the new determinant is $(-1)^{|N_n|} |\hat{f}(N_i \cap N_j)|_{i,j=1}^{n} = f(N_n) |\hat{f}(N_i \cap N_j)|_{i,j=1}^{n-1}$. If we note that $N_1 = \varnothing$ and $\hat{f}(\varnothing) = f(\varnothing)$, the theorem follows by induction.

EXAMPLE. Let $f(N) = 2^{|N|}$. Then $\hat{f}(M) = (-1)^{|M|}$. It follows that $2^{\alpha(\mathfrak{F})}$ equals a determinant with all entries $+1$ or -1. If $\mathfrak{F}$ is the family which corresponds to the integers $0, 1, \cdots, n$, we get Theorem 1 in [2].

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