

TOPOLOGICAL GROUPS WITH EQUAL LEFT AND RIGHT UNIFORMITIES¹

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If (G, σ) is a topological group with topology σ , there is a natural topology τ on G such that $\tau \supset \sigma$, (G, τ) is a topological group with equal left and right uniformities and τ is the smallest such topology. The topology τ can be obtained as follows. If \mathcal{V} is a neighborhood base at the identity in $(G, \sigma) = G$, then $\{\bigcap_{t \in G} tVt^{-1} \mid V \in \mathcal{V}\}$ is a neighborhood base at the identity in $(G, \tau) = G^*$. Here we are interested in determining the nature of G^* given G and also in obtaining results on groups with equal left and right uniformities. For results on such groups we point to Braconnier [1] in which it is shown that a locally compact group with equal left and right uniformities has a two sided invariant Haar measure, and to N. Rickert's unpublished theorems on the structure of locally compact groups with equal uniformities. In contrast, T. S. Wu [10] has shown that left almost periodic functions on locally compact groups need not be right almost periodic. Of course, the groups here must have distinct left and right uniformities. These groups were discovered at an intermediate stage in the construction of a locally compact group G for which G^* is not locally compact. This example will be described later.

One of our principal results (Theorem 1) is that G^* is a Lie group if G is a connected Lie group. Since this depends only on the fact that G^* is locally compact, we have attempted to determine more precisely when this occurs. As mentioned above, there are locally compact groups G for which G^* is not locally compact; however, using the result above on Lie groups we show that, if G is locally compact and connected, then G^* is locally compact. It is also shown that if a connected locally compact group G with equal left and right uniformities contains a one parameter subgroup $\phi(R)$ as a normal subgroup, then $\phi(R)$ is contained in the center of G . The hypothesis on the uniformities is not necessary if $G/\phi(R)$ is compact. These are Corollaries 1 and 2 of Theorem 3.

As a preliminary we obtain some lemmas on the topology of a subgroup $H \subset G$ relative to G^* .

LEMMA 1. *If K is a compact normal subgroup of the connected group G , then the topologies of K relative to G and G^* are equal.*

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PROOF. Let $A(K)$ be the group of all continuous automorphisms of K and $I(K)$ the group of inner automorphisms of K . Define $\Delta: G \rightarrow A(K)$ as follows: $\Delta(g)(k) = gkg^{-1}$. We note that Δ is continuous when $A(K)$ has the compact-open = uniform topology. Thus $\Delta(G)$ is a connected subgroup of $A(K)$. It follows that $\Delta(G) = I(K)$ since $A(K)/I(K)$ is totally disconnected by Theorem 1 of [6]. Let U be a neighborhood of the identity in G . By compactness of K we pick a neighborhood V such that $V \subset \bigcap_{k \in K} kUk^{-1}$. We have,

$$K \cap \left(\bigcap_{t \in G} tUt^{-1} \right) = \bigcap_{t \in G} t(U \cap K)t^{-1} = \bigcap_{k \in K} k(U \cap K)k^{-1} \supset V \cap K.$$

This proves the lemma.

LEMMA 2. *If H is a subgroup of G such that the coset space G/H is compact, then the sets $\bigcap_{h \in H} hVh^{-1}$, where V is a neighborhood of e in G , form a neighborhood base at e in G^* .*

PROOF. Let V be a neighborhood of e in G and let W be a symmetric neighborhood of e such that $W^3 \subset V$. Since G/H is compact, $G = \bigcup_{i=1}^n Wg_iH$ for some sequence g_1, \dots, g_n in G . Now

$$V_i = \bigcap_{t \in Wg_iH} t^{-1}Vt \supset \bigcap_{h \in H} h^{-1}g_i^{-1}Wg_ih$$

and

$$\bigcap_{t \in G} t^{-1}Vt = \bigcap_{i=1}^n V_i \supset \bigcap_{h \in H} h^{-1} \left(\bigcap_{i=1}^n g_i^{-1}Wg_i \right) h.$$

This proves the lemma.

COROLLARY. *If in addition to the hypothesis of Lemma 2, H has equal left and right uniformities, then the topologies of H relative to G and G^* are equal.*

LEMMA 3. *Let G be a topological group such that $G=LK$ where K is a compact invariant subgroup and L is a connected subgroup. Suppose that, if V is a neighborhood of the identity relative to L , then VK is a neighborhood of the identity in G and for some neighborhood V_0 of e relative to L , $V_0 \cup K = e$. Then $G^*/K = (G/K)^*$.*

PROOF. Let U be a neighborhood of the identity in G and let V be a symmetric neighborhood of the identity relative to L such that $V \subset U$ and $V^6 \subset V_0$. Let W be a neighborhood of e in G such that $WK \subset VK$. Since K is compact, we can choose a neighborhood N such that $kN \subset Wk$ for each $k \in K$. To prove the lemma it is sufficient

to show that $\bigcap_{g \in G} gNKg^{-1} \subset (\bigcap_{l \in L} lUl^{-1})K$ since the second set is a neighborhood of the identity in G^*/K by Lemma 2. Now $\bigcap_{g \in G} gNKg^{-1} = \bigcap_{l \in L, k \in K} lkNKl^{-1} \subset \bigcap_{l \in L} lWKl^{-1} \subset \bigcap_{l \in L} lVKl^{-1}$. We complete the proof by showing

$$(\alpha) \quad \bigcap_{l \in L} lVKl^{-1} = \left(\bigcap_{l \in L} lVl^{-1} \right) K.$$

Since L is connected, we have $\bigcap_{n=1}^{\infty} \bigcap_{l \in V^n} lVKl^{-1} = \bigcap_{l \in L} lVKl^{-1}$. Let $x \in \bigcap_{l \in V} lVKl^{-1} = \bigcap_{l \in V} lVl^{-1}K$. Then for each $l \in V$ we have $x \in lVl^{-1}k$. To see that k does not depend on l let $x = lv^{-1}k_1 = hv'h^{-1}k_2$ where l, v, h, v' are in V and k_1, k_2 are in K . Then, since $V_0 \cap K = e$ and $V^6 \subset V_0$, it follows that $k_1 = k_2$ and $x \in (\bigcap_{l \in V} lVl^{-1})K$. By induction we have $\bigcap_{l \in V^n} lVl^{-1}K = (\bigcap_{l \in V^n} lVl^{-1})K$ and a simple argument now shows that (α) is satisfied.

We will now make use of the following theorem proved by Iwasawa and Yamabe. (See Mem. Amer. Math. Soc., No. 14, 1955; p. 24.)

THEOREM. *Let G be a locally compact connected group. There is a neighborhood N of the identity of G , a closed local Lie group L_0 in N and a compact subgroup K such that $N = L_0 \times K$.*

If we let L be the group generated by L_0 , then using the method of the proof of Lemma 3 we obtain the conclusion of that lemma in this case and G/K is a Lie group. To see this let V be a neighborhood of the identity in L_0 such that $V^3 \subset L_0$. Now assume that VK , as a neighborhood of the identity of G/K , contains a nontrivial subgroup. Then the inverse image H of this subgroup properly contains K and a simple computation shows that $H \cap V$ is a nontrivial group contained in L_0 which is a contradiction. We have proved the following.

LEMMA 4. *If G is a locally compact connected group, then there is a compact invariant subgroup K of G such that G/K is a Lie group and $G^*/K = (G/K)^*$.*

THEOREM 1. *If G is a connected Lie group, then G^* is a Lie group.*

PROOF. It is sufficient to prove that G^* is locally compact. To do this we use 1.12, p. 100 of [4]. Let M be the tangent space of G at e . For $g \in G$ let ϕ_g be the inner automorphism determined by g and denote by $d\phi_g$ the differential of ϕ_g . Now $d\phi_g$ is an isomorphism of M and $\exp d\phi_g = \phi_g \exp$, where \exp is the exponential map of M to G . Now let W be a neighborhood of the origin in M such that $\exp|_W$ is a homeomorphism. There are neighborhoods U and V of e in G such that $V = \exp Z$, where Z is symmetric and convex and

$gVg^{-1} \subset \exp W$ for $g \in U$. Since $g(\exp Z)g^{-1} \subset \exp W$, $d\phi_g(Z) \subset W$ for $g \in U$. We obtain a sequence of sets $\{V_n\}$ as follows:

$$V_1 = \bigcap_{g \in U} gVg^{-1},$$

$$V_n = \bigcap_{g \in U} gV_{n-1}g^{-1} = \bigcap_{g \in U^n} gVg^{-1}.$$

Let $F(X) = \bigcap_{g \in U} d\phi_g(X)$. We show

(α) $\exp F^n(Z) = V_n.$

First $\exp F(Z) = \exp \bigcap_{g \in U} d\phi_g(Z) = \bigcap_{g \in U} \exp d\phi_g(Z)$ since \exp is 1-1 on $d\phi_g(Z)$. Thus, $\exp F(Z) = \bigcap_{g \in U} \phi_g(\exp Z) = V_1$. This proves (α) for $n = 1$ and since $V_1 \subset V$ we have also shown that $F(Z) \subset Z$ and hence $F^n(Z) \subset Z$. It follows that \exp is 1-1 on $d\phi_g(F^n(Z))$. Now suppose that (α) holds for the positive integer n . Then $\exp F^{n+1}(Z) = \exp \bigcap_{g \in U} d\phi_g(F^n(Z)) = \bigcap_{g \in U} \phi_g(V_n) = V_{n+1}$ and (α) is proved. Since Z is symmetric and convex $\bigcap F^n(Z)$ is symmetric and convex. Thus, $\exp \bigcap F^n(Z) = \bigcap V_n$ contains a Euclidean neighborhood of e . But $\bigcap_n V_n = \bigcap_n \bigcap_{g \in U^n} gVg^{-1} = \bigcap_{g \in G} gVg^{-1}$. This proves the theorem.

THEOREM 2. *If G is a locally compact connected group, then G^* is locally compact.*

PROOF. By Lemma 4 pick a compact invariant subgroup K such that G/K is a Lie group and $G^*/K = (G/K)^*$. It follows that G^*/K is locally compact by Theorem 1. By Lemma 1 K is a compact subset of G^* and by a theorem of Gleason [3, 1.10] G^* is locally compact.

The following is an example of a locally compact group G such that G^* is not locally compact.

EXAMPLE. Let $A_i = Z_2$, the discrete two element group, for each integer i . Let $H_0 = \pi_i A_i$ and let Z be discrete infinite cyclic with generator z . Define an operation in $G_0 = Z \times H_0$ as follows, $(z^n, x)(z^m, y) = (z^{n+m}, x_{-m}y)$ where the i th coordinate of x_{-m} is the $(i-m)$ th coordinate of x . Thus G_0 is a semidirect product [5]. It is easy to see that G_0 is a locally compact group with this operation and the usual product topology. Let $G_i = G_0$ and $H_i = \{e\} \times H_0$ for $i = 1, 2, \dots$. Let $H = \prod H_i$ and $G = \prod G_i$ with coordinatewise operation, the usual product topology on H and H open in G . Thus, G is a locally compact group and H is a compact open subgroup of G . Let V_i be a proper neighborhood of e in H_i . If $U_n = V_1 \times \dots \times V_n \times (\prod_{i>n} H_i)$, then $\bigcap_{i \in G} iU_n t^{-1} = \{e\} \times \dots \times \{e\} \times (\prod_{i>n} H_i)$. Since $\{U_n\}$ is a neighborhood base at the identity of G , it follows that G^* is not locally compact.

A more general construction than that above is obtained by letting F be a compact infinite group which admits an expansive automorphism ϕ . Let H be the discrete group generated by ϕ . Now replace G_0 by the semidirect product $F \circledast H$ in the above example. The existence of expansive automorphisms on certain compact groups has been established by R. F. Williams [3], T. S. Wu [11], M. Eisenberg [2] and F. Reddy [8].

By a one parameter subgroup in a group G we mean the image $\phi(R)$ of the reals under a continuous homomorphism $\phi: R \rightarrow G$.

THEOREM 3. *If G is a connected locally compact group and $\phi(R)$ is a one parameter subgroup of G^* which is normal, then $\phi(R)$ is contained in the center of G .*

PROOF. By Theorem 2 G^* is locally compact. Thus ϕ is a topological isomorphism or $\bar{\phi}(R)$ is compact in G^* [7, p. 102]. In case $\bar{\phi}(R)$ is compact in G^* it is also compact in G and hence contained in the center of G [5, 26.10]. Now suppose that ϕ is an isomorphism. Pick a neighborhood V of e in G such that $\bigcap_{g \in G} g V g^{-1} \cap \phi(R) = (-\epsilon, \epsilon)$. Let α be in $(0, \epsilon)$ and define a sequence in $(-\epsilon, \epsilon)$ as follows, $\alpha_1 = g\alpha g^{-1}$, $\alpha_n = g\alpha_{n-1}g^{-1}$ where g is any element of G . The sequence $\{\alpha_n\}$ is monotone. Let $\beta = \lim \alpha_n$. It follows that $g\beta g^{-1} = \beta$. We can assume that β is nonzero since, if $\{\alpha_n\}$ is decreasing, on replacing g by g^{-1} we get an increasing sequence and its limit will also commute with g . It is easy to show that if g commutes with a nonzero real number β then it commutes with every rational multiple of β hence with every real number. Thus, $\phi(R)$ is contained in the center of G .

As an immediate consequence of Theorem 3 we have.

COROLLARY 1. *If G is a connected locally compact group with equal left and right uniformities which contains a one parameter group as a normal subgroup then this subgroup is in the center of G .*

COROLLARY 2. *If G is a connected locally compact group which contains a one parameter normal subgroup $\phi(R)$ such that $G/\bar{\phi}(R)$ is compact, then $\phi(R)$ is contained in the center of G .*

PROOF. If $\phi(R)$ is compact, then the conclusion follows from [5, 26.10]. Otherwise ϕ is an isomorphism [7, p. 102] and by the Corollary of Lemma 2 $\phi(R)$ is a one parameter subgroup of G^* . The corollary now follows from the theorem.

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