MEASURES THAT VANISH ON HALF SPACES[1]

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I. **Introduction.** It is well known that if $f \in L^1(E_n)$ (E_n denotes real Euclidean n-space, and all functions are complex valued) has the property that

$$\int_H f(x)dx = 0 \tag{1}$$

for all half spaces H, then $f(x) = 0$ a.e. It is natural to conjecture that if (1) holds for all $H \in \mathcal{H}_n$, where $\mathcal{H}_n$ is the set of all half spaces of E_n that exclude the unit sphere, then $f(x) = 0$ a.e. in $\{|x| = 1\}$. Recently S. Helgason has proven this assuming the a priori estimate $f(x) = O(|x|^{-m})$ for all $m > 0$ [**5**]. The simple example (due to D. J. Newman) of $f(x) = 1/(x_1 + ix_2)^3$ if $|x| \geqq 1$ and zero otherwise, which by Cauchy's theorem satisfies (1) for all $H \in \mathcal{H}_2$, shows that without some assumption the conjecture is in fact false.

The purpose of this note is to characterize explicitly those $f \in L^1(E_n)$ that satisfy (1) for all $H \in \mathcal{H}_n$. The second section is devoted to a Paley-Wiener theorem for Hankel transforms which is needed in the proof of the main result. This is found in the final section together with a few concluding remarks. Another version of a Paley-Wiener theorem for Hankel transforms may be found in [**4**]. I am indebted to the referee for this reference.

II. **A P-W Theorem for Hankel transforms.** The following theorem of Plancherel and Polya [**7**] will be used.

(P) *If* $f \in L^1(E_n)$ *and* $F(y) = \int_{E_n} f(x) \exp(-2\pi i(x, y))dx$ *(or* $f \in L^2(E_n)$ *and* F *its Fourier transform) then* f *vanishes a.e. in* $\{|x| \geqq 1\}$ *if and only if* F *is an entire function of exponential type* 2π *in every direction.*

The Hankel transform of order ν is defined by

$$F(y) = \int_0^\infty f(x) J_\nu(2\pi xy)(xy)^{1/2} dx \tag{2}$$

where J_ν is the Bessel function of order ν and either $f \in L^1(0, \infty)$ or $f \in L^2(0, \infty)$ and the integral is taken as l.i.m. [**2**, §42].

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LEMMA 1. *If F is given by* (2) *and* $\nu=n-\frac{1}{2}$ $(n=1, 2, \cdots)$ (*resp.* $\nu=n$) *then F (resp. $y^{1/2}F(y)$) is an entire function of exponential type 2π if and only if f (resp. $x^{-1/2}f(x)$) $=f_0+\sum_{j=1}^{n} c_j r_j$ where $f_0(x)=0$ for $x\geqq 1$, $r_j(x)=x^{-j}$ if $x\geqq 1$ and zero otherwise, and the c_j's are constants.*

PROOF. The proof for $\nu=n$ is just like the proof for $\nu=n-1/2$ and so we confine our attention to this latter case. Furthermore, it is quite straightforward to reduce the case of $f\in L^1$ to $f\in L^2$ and thus it suffices to prove the lemma assuming that $f\in L^2$.

(a) Suppose first that F has a zero of order $\geqq n$ at $y>0$. Then $G(y)=F(y)/y^n$ is also an entire function of exponential type 2π and setting $g(x)=f(x)/x^n$ we have

$$G(y) = y^{-n+1/2}\int_0^\infty g(x)J_\nu(2\pi xy)x^{n+1/2}dx. \tag{3}$$

If g is considered as a radial function in $L^2(E_{2n+1})$ then except for a factor of 2π the right hand side of (3) gives the Fourier transform of g [1, §2.6]. Applying (P) we conclude that in this case f itself vanishes for $x\geqq 1$.

(b) If r_j is inserted for f in (2) the corresponding R_j are given by

$$R_j = y^{j-1}\int_0^\infty J_\nu(2\pi x)x^{-j+1/2}dx + S_j$$

where S_j is an entire function of exponential type 2π and vanishes at zero to order $\geqq n$. From (a) we see that the first term cannot vanish and hence if a suitable linear combination of r_j $(j=1, 2, \cdots, n)$ is subtracted from f we are reduced to the situation in (a), and this completes the proof in one direction. The other direction is immediate since J_ν is entire of exponential type 2π, a fact which we used in our assertion about S_j.

The following is an immediate consequence of the lemma and (P).

COROLLARY. *If $f\in L^1(E_n)$ is a radial function $(f(x)=\tilde{f}(|x|))$ and*

$$\tilde{F}(y) = 2\pi i^k y^{-n}\int_0^\infty \tilde{f}(w/y)J_{k+(n-1)/2}(2\pi w)w^{n/2}dw,$$

then F is entire of exponential type 2π if and only if $f=f_0+\sum_{j=1}^{k-1} c_j r_j$ $(n\geqq 2,\ k>0)$.

III. **The main theorem.** Let

$$A_n=\{f\epsilon L^1(E_n): (1) \text{ holds for all } H\in\mathcal{H}_n\}.$$

Observe that (i) A_n is a closed subspace of $L^1(E_n)$, and (ii) A_n is rotation invariant in the sense that $R \in SO(n)$ and $f \in A_n$ implies that $Rf \in A_n(Rf(x)=f(Rx))$. Denote by $\sum f_k(|x|, x')$ $(x'=x/|x|)$ the expansion of f in spherical harmonics, i.e., for fixed $|x|$, f_k is a spherical harmonic of degree k [1, §2.7]. The main reduction is accomplished by

LEMMA 2. *If $f \in L^1(E_n)$ then $f \in A_n$ if and only if $f_k \in A_n$ for all k.*

PROOF. Assume first that $f \in A_n$, then f_k may be expressed as

$$f_k(|x|, x') = \int_{SO(n)} Rf(x) Z_k(Ry', y') dR \tag{4}$$

where y' is a fixed unit vector and Z_k is a zonal harmonic of degree k [3, XI]. Since translation is continuous in the L^1 norm, one verifies easily that $R \to Rf$ is a continuous map from $SO(n) \to L^1(E_n)$ and it then follows from (i) and (ii) that $f_k \in A_n$.

Conversely, if $f_k \in A_n$ for all k then so are the appropriate Abel means of $\sum f_k$. Now the Abel means of a continuous function converge [6][2] and thus by (i) $f \in A_n$.

Next we identify the Fourier transform of A_n in

LEMMA 3. *If $f \in L^1(E_n)$ and $F(y) = \int_{E_n} f(x) \exp(-2\pi i(y, x))dx$ then $f \in A_n$ if and only if for all $t \in E_n$ with $|t|=1$ we have that F_t is an entire function of exponential type 2π where $F_t(z) = F(t_1 z, t_2 z, \cdots, t_n z)$.*

PROOF. By (ii) it suffices to consider $t=(1, 0, \cdots, 0)$. Fubini's theorem yields

$$\begin{aligned} F_t(z) &= \int_{E_n} f(x) \exp(-2\pi i z x_1) dx \\ &= \int_{E_1} \exp(-2\pi i z x_1) \left\{ \int_{E_{n-1}} f(x) dx_2 \cdots dx_n \right\} dx_1. \end{aligned} \tag{5}$$

Since $F \in A_n$, $\{\ \}$ as a function of x_1 vanishes for $|x_1| \geqq 1$ and thus F_t is entire of exponential type 2π. The converse follows from (5) and (P).

THEOREM. *If $f \in L^1(E_n)$ and f_k are given by* (4) *then $f \in A_n$ if and only if*

$$f_k(|x|, x') = f_{k,0}(|x|, x') + \sum_{j=1}^{k-1} c_{k,j}(x') r_{k+j}(|x|)$$

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[2] The theorem is given there for real valued continuous functions but may be easily extended to continuous functions with values in a Banach space (here L^1).

where $f_{k,0}$ vanishes for $|x| \geqq 1$. $c_{k,j}$ are harmonics of degree k (as is $f_{k,0}$ for fixed $|x|$) and r_m is defined in Lemma 1.

PROOF. If F_k denotes the Fourier transform of f_k then

$$F_k(|y|, y') = 2\pi i^k |y|^{-n} \int_0^\infty f_k(w/|y|, y') J_{k+(n-1)/2}(2\pi w) w^{n/2} dw$$

[2, §2.7]. The theorem now follows from Lemma 2-3 and the corollary of Lemma 1.

Helgason's result is obtained upon noticing that his a priori bound on f carries over to f_k and implies that the $c_{k,j}$ are identically zero. The results of this note carry over *mutatis mutandis* to measures, the details are omitted.

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BIBLIOGRAPHY

1. S. Bochner, *Harmonic analysis and the theory of probability*, Univ. of California Press, Berkeley and Los Angeles, Calif., 1960.

2. ———, *Lectures on Fourier integrals*, Annals of Mathematics Studies No. 42, Princeton Univ. Press, Princeton, N. J., 1959.

3. A. Erdelyi, et al., *Higher transcendental functions*, Vol. 2, McGraw-Hill, New York, 1953.

4. J. L. Griffith, *Hankel transforms of functions zero outside a finite interval*, J. Proc. Roy. Soc. New South Wales **89** (1956), 109–115.

5. S. Helgason, *The Randon transform on Euclidean spaces, compact two-point homogenous spaces and Grassman manifolds*, Acta Math. **113** (1965), 153–180.

6. T. Kubota, *Ueber die Konvex-geschlossenen Mannigfaltigkeiten in n-dimensionale Raume*, Sci. Rep. Tôhoku Imperial Univ. **14** (1925), 85–99.

7. M. Plancherel and G. Polya, *Fonctions entières et integral de Fourier multiples*, Comment Math. Helv. **9** (1936–1937), 224–248.

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