REPRESENTATION OF FUNCTIONS IN \( C(X) \)
BY MEANS OF EXTREME POINTS

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Let \( X \) be a compact metric space. It is known that if \( U \) is the closed unit ball of \( C_r(X) \) (the space of continuous real-valued functions on \( X \) under the usual sup norm), a necessary and sufficient condition that \( U \) be the closed convex hull of the set of its extreme points is that \( X \) be totally disconnected (Bade \[1\]). It is also known (Phelps \[4\]) that if \( C(X) \) is the space of all continuous complex-valued functions on \( X \) under the sup norm, and if \( U \) is the closed unit ball of \( C(X) \), \( U \) is always equal to the closed convex hull of the set of its extreme points (see also Goodner \[2\]). It is our purpose in this note to obtain information about \( U \) in the case of \( C(X) \) similar to that obtained for \( C_r(X) \).

We make the following notational conventions: \( D \) will denote the closed unit disc in the complex plane and \( B \) will denote the set of points in \( D \) of modulus 1. By \( E \) we will mean the set of extreme points of \( U \) (the closed unit ball of \( C(X) \)); \( E \) is the set of all elements of \( U \) which map \( X \) into \( B \). The topological dimension of \( X \) as defined in Hurewicz and Wallman \[3\] will be denoted by \( \dim X \).

Our theorem now reads as follows:

**Theorem.** Let \( X \) be a compact metric space. Then the following are equivalent:

1. \( \dim X \leq 1 \);
2. \( U \) is a subset of the convex hull of \( E \).

**Proof.** We first observe that if \( f \) is a continuous map of a topological space \( Y \) into \( D \) which omits the origin, then there are two continuous maps \( f_1 \) and \( f_2 \) of \( Y \) into \( B \) such that \( f = (f_1 + f_2)/2 \). We now show that condition (1) implies condition (2). (I am indebted to the referee for strengthening and combining several arguments to give the following proof.)

Let \( f \) be in \( U \). By Theorem VI.1 of Hurewicz and Wallman, the origin is an unstable value of \( f \); by Proposition B of the same section, there is a continuous function \( h_1 \) which omits the origin such that

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If $|f(x)| \geq 1/3$, then $h_1(x) = f(x)$;

If $|f(x)| < 1/3$, then $|h_1(x)| < 1/3$.

Put $h_2 = 2f - h_1$. Then it is clear that $h_1$ and $h_2$ are in $U$.

Suppose $|h_1(x)| > 3\epsilon > 0$ for all $x \in X$. By the same results in [3], there is a continuous function $g_2$ such that $g_2$ omits the origin and such that

If $|h_2(x)| \geq \epsilon$, then $g_2(x) = h_2(x)$;

if $|h_2(x)| < \epsilon$, then $|g_2(x)| < \epsilon$.

Put $g_1 = 2f - g_2$. Now it is easy to check that $g_1$ and $g_2$ are in $U$; moreover $g_1$ omits the origin since $|g_1(x) - h_1(x)| = |g_2(x) - h_2(x)| \leq 2\epsilon$ for all $x \in X$. By the remark at the beginning of the proof, $g_1$ and $g_2$ are in the convex hull of $E$; hence $f = (g_1 + g_2)/2$ is in the convex hull of $E$.

We now prove that condition (2) implies condition (1). By [3, Theorem VI, §4] it suffices to prove the following: Let $C$ be a closed subset of $X$. Then if $f$ is a continuous map of $C$ into $B$, there is an extension of $f$ to a continuous map of $X$ into $B$.

Hence, let $C$ and $f$ be as above. Using Tietze's theorem, we can extend $f$ to a continuous $\tilde{f}$ from $X$ into $D$. If condition (2) holds, there is a probability measure $\mu$ on $U$ (even one with finite support) such that $\mu(E) = 1$ and such that $L(\tilde{f}) = \int L(g)d\mu(g)$ for all $L$ in the (complex) dual of $C(X)$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence dense in $C$ and define linear functionals $L_n$ on $C(X)$ by $L_n(h) = h(x_n)$ for $h$ in $C(X)$. Then for each $n$ we have

$$\tilde{f}(x_n) = L_n(\tilde{f}) = \int L_n(g)d\mu(g) = \int g(x_n)d\mu(g);$$

we may divide to obtain

$$1 = \int_{E} \frac{g(x_n)}{\tilde{f}(x_n)} d\mu(g) \quad \text{for all } n.$$

Since $|\tilde{f}(x_n)| = |g(x_n)| = 1$ for all $g$ in $E$ and since $\mu$ is a probability measure, it must be the case that

$$\mu\{g \in E: g(x_n) \neq \tilde{f}(x_n)\} = 0 \quad \text{for each } n.$$

Hence,

$$\mu\left(\bigcup_{n=1}^{\infty} \{g \in E: g(x_n) \neq \tilde{f}(x_n)\}\right) = 0;$$
it follows that there is a $g^*$ in $E$ such that $g^*(x_n) = f(x_n) = f(x_n)$ for all $n$; since $\{x_n\}$ is dense in $C$, $g^*(x) = f(x)$ for all $x$ in $C$. This $g^*$ is the desired extension of $f$ and the proof is thereby complete.

**Bibliography**


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