REPRESENTATION OF FUNCTIONS IN $C(X)$ BY MEANS OF EXTREME POINTS

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Let X be a compact metric space. It is known that if U is the closed unit ball of $C_r(X)$ (the space of continuous real-valued functions on X under the usual sup norm), a necessary and sufficient condition that U be the closed convex hull of the set of its extreme points is that X be totally disconnected (Bade [1]). It is also known (Phelps [4]) that if $C(X)$ is the space of all continuous complex-valued functions on X under the sup norm, and if U is the closed unit ball of $C(X)$, U is always equal to the closed convex hull of the set of its extreme points (see also Goodner [2]). It is our purpose in this note to obtain information about U in the case of $C(X)$ similar to that obtained for $C_r(X)$.

We make the following notational conventions: D will denote the closed unit disc in the complex plane and B will denote the set of points in D of modulus 1. By E we will mean the set of extreme points of U (the closed unit ball of $C(X)$); E is the set of all elements of U which map X into B. The topological dimension of X as defined in Hurewicz and Wallman [3] will be denoted by dim X.

Our theorem now reads as follows:

THEOREM. *Let X be a compact metric space. Then the following are equivalent*:

(1) dim $X \leqq 1$;

(2) *U is a subset of the convex hull of E.*

PROOF. We first observe that if f is a continuous map of a topological space Y into D which omits the origin, then there are two continuous maps f_1 and f_2 of Y into B such that $f=(f_1+f_2)/2$. We now show that condition (1) implies condition (2). (I am indebted to the referee for strengthening and combining several arguments to give the following proof.)

Let f be in U. By Theorem VI.1 of Hurewicz and Wallman, the origin is an unstable value of f; by Proposition B of the same section, there is a continuous function h_1 which omits the origin such that

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(1) If $|f(x)| \geqq 1/3$, then $h_1(x) = f(x)$;

(2) if $|f(x)| < 1/3$, then $|h_1(x)| < 1/3$.

Put $h_2 = 2f - h_1$. Then it is clear that h_1 and h_2 are in U.

Suppose $|h_1(x)| > 3\epsilon > 0$ for all $x \in X$. By the same results in [3], there is a continuous function g_2 such that g_2 omits the origin and such that

(3) If $|h_2(x)| \geqq \epsilon$, then $g_2(x) = h_2(x)$;

(4) if $|h_2(x)| < \epsilon$, then $|g_2(x)| < \epsilon$.

Put $g_1 = 2f - g_2$. Now it is easy to check that g_1 and g_2 are in U; moreover g_1 omits the origin since $|g_1(x) - h_1(x)| = |g_2(x) - h_2(x)| \leqq 2\epsilon$ for all $x \in X$. By the remark at the beginning of the proof, g_1 and g_2 are in the convex hull of E; hence $f = (g_1 + g_2)/2$ is in the convex hull of E.

We now prove that condition (2) implies condition (1). By [3, Theorem VI, §4] it suffices to prove the following: Let C be a closed subset of X. Then if f is a continuous map of C into B, there is an extension of f to a continuous map of X into B.

Hence, let C and f be as above. Using Tietze's theorem, we can extend f to a continuous $\tilde{f}$ from X into D. If condition (2) holds, there is a probability measure μ on U (even one with finite support) such that $\mu(E) = 1$ and such that $L(\tilde{f}) = \int L(g)d\mu(g)$ for all L in the (complex) dual of $C(X)$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence dense in C and define linear functionals L_n on $C(X)$ by $L_n(h) = h(x_n)$ for h in $C(X)$. Then for each n we have

$$\tilde{f}(x_n) = L_n(\tilde{f}) = \int L_n(g)d\mu(g) = \int g(x_n)d\mu(g);$$

we may divide to obtain

$$1 = \int_E \frac{g(x_n)}{\tilde{f}(x_n)}\, d\mu(g) \qquad \text{for all } n.$$

Since $|\tilde{f}(x_n)| = |g(x_n)| = 1$ for all g in E and since μ is a probability measure, it must be the case that

$$\mu\{g \in E: g(x_n) \neq \tilde{f}(x_n)\} = 0 \qquad \text{for each } n.$$

Hence,

$$\mu\left(\bigcup_{n=1}^{\infty} \{g \in E: g(x_n) \neq \tilde{f}(x_n)\}\right) = 0;$$

it follows that there is a g^* in E such that $g^*(x_n) = \bar{f}(x_n) = f(x_n)$ for all n; since $\{x_n\}$ is dense in C, $g^*(x) = f(x)$ for all x in C. This g^* is the desired extension of f and the proof is thereby complete.

Bibliography

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