ON FILIPPOV'S IMPLICIT FUNCTIONS LEMMA[1]

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In 1959 A. F. Filippov published a paper [1] containing a lemma designed for use in the study of optimal control problems. Stated somewhat imprecisely, let k be a continuous function on a compact set Q in a finite dimensional space and with values in a finite dimensional space, and let $(u'(t)\colon a \leqq t \leqq b)$ be a function with values in Q such that $k(u'(\cdot))$ is measurable; then there exists a measurable function u from $[a, b]$ to Q such that $k(u(t)) = k(u'(t))$.

For purposes of stochastic control theory it is desirable to extend this to allow arbitrary measure spaces, instead of intervals of real numbers (Kushner [**4**]); for purposes of the calculus of variations it is desirable to relax the requirement of compactness on Q. We do both of these (Q may be any separable metric space), and at the same time we permit the values of k to lie in any Hausdorff space; it costs nothing. In §2 we give an application to optimal control theory.

ADDED IN PROOF. In an abstract in Amer. Math. Monthly **73** (1966), p. 927, M. Q. Jacobs announces a generalization of Filippov's lemma. This generalization is a special case of our Theorem 1.

1. Definitions and first theorem.

If $\mathfrak{M}$ is a σ-ring of subsets of a set M, and S is a topological space, a function $g\colon M_0 \to S$ from a set M_0 of the class $\mathfrak{M}$ to S is called measurable if the inverse image of every compact set in S belongs to $\mathfrak{M}$.

THEOREM 1. *Let M be a measure space, A a Hausdorff space, and Q a topological space which is the union of a countable number of compact metrizable subsets. Let $k\colon Q \to A$ be continuous, and $y\colon M \to A$ a measurable function such that $y(M) \subseteq k(Q)$. Then there exists a measurable function $u\colon M \to Q$ such that*

$$k(u(x)) = y(x) \quad \textit{for all } x \textit{ in } M.$$

PROOF. We proceed through a number of special cases.

Case 1. Let Q be a closed subset of the right half line $[0, \infty)$. This set Q we also call L, for convenience later, and we construct a measurable function $T\colon M \to L$ such that $k \circ T = y$.

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For each nonnegative integer q we partition $k(L)$ into a disjoint union of differences of compact sets, B_j^q, $j=1, 2, \cdots$. We define the sets B_j^q as follows:

$$B_j^q = k(L \cap [0, j \cdot 2^{-q}]) - k(L \cap [0, (j-1) \cdot 2^{-q}]).$$

For each $x \in M$ and each nonnegative integer q, set

$$T_q(x) = \inf k^{-1}(B_j^q),$$

where j is that integer for which $y(x) \in B_j^q$. This function is trivially measurable, since it takes on only the countable set of values $\inf k^{-1}(B_j^q)$, $j=1, 2, \cdots$, and the inverse images of these sets B_j^q are measurable.

We claim now that the T_q are an increasing sequence converging to our desired function T.

The sequence is increasing since for each x and each q there are integers i, j such that $y(x)$ belongs to B_j^q and B_i^{q+1}; in fact we have $i=2j-1$ or $i=2j$. By definition

$$B_i^{q+1} \subseteq B_j^q. \tag{1}$$

Since $T_{q+1}(x) = \inf k^{-1}(B_i^{q+1})$ and $T_q(x) = \inf k^{-1}(B_j^q)$ we have $T_{q+1}(x) \geqq T_q(x)$.

For each x, the sequence $T_q(x)$ is bounded above, for if $x \in k^{-1}(B_j^0)$ then $T_q(x) \leqq j$ for all q. Hence T_q converges to a measurable function T with values in L (since L is closed).

We finally claim that $k \circ T = y$. If this is false, then for some $x \in M$ there exists an open subset U of A such that $k(T(x)) \in U$, $y(x) \notin U$. Since k is continuous, $k^{-1}(U)$ contains some neighborhood of $T(x)$. Therefore there is some q and some j such that

$$T(x) \in L \cap ((j-1) \cdot 2^{-q}, j \cdot 2^{-q}]$$

and

$$L \cap [(j-1) \cdot 2^{-q}, j \cdot 2^{-q}] \subseteq k^{-1}(U).$$

This implies that $T_q(x) = \inf k^{-1}(B_j^q)$ for this j. (Otherwise, by (1), we would have for all $n \geqq q$, $T_n(x) \leqq (j-1) \cdot 2^{-q}$, while $\lim_{n \to \infty} T_n(x) = T(x) > (j-1)2^{-q}$.) Since for this j, $B_i^q \subseteq U$, we have, by the definition of T_q, that $y(x) \in B_j^q \subseteq U$, giving a contradiction. This completes the proof of Case 1.

Case 2. We now let Q be any space such that there is a closed subset L of $[0, \infty)$ and a continuous map $\phi: L \to Q$, taking L onto Q. We then have the following picture:

$$L \xrightarrow{\phi} Q \xrightarrow{k} A \xleftarrow{y} M,$$

where k and ϕ are continuous and y is measurable, and $y(M) \subseteq k(Q) = k(\phi(L))$. By Case 1, there is a measurable function $T: M \to L$ so that $(k \circ \phi) \circ T = y$. We set $u = \phi \circ T$ and claim that this is our desired function. We have $y = k \circ u$ immediately, since $y = k \circ \phi \circ T$. u is also measurable for, if C is a compact subset of Q, $\phi^{-1}(C)$ is closed in L and hence is a countable union of compact subsets of L (namely the sets $L \cap [0, n] \cap \phi^{-1}(C)$). $T^{-1}(\phi^{-1}(C))$ is therefore measurable, and this is exactly the desired statement that $u^{-1}(C)$ is measurable.

Case 3. We now prove the theorem as stated. Let $K_1, K_2, \cdots$ be a sequence of compact metrizable sets whose union is Q. Since every compact metric space is the continuous image of the Cantor discontinuum [**2**, Theorem 3.28], for each positive integer n there is a closed subset L_n of $[2n-1, 2n]$ (a translate of the Cantor set) and a continuous function $\phi_n: L_n \to K_n$ whose range is K_n. Define $L = \bigcup L_n$, and define ϕ to be the function on L which coincides with ϕ_n on L_n. Now the hypotheses of Case 2 are satisfied and the proof is complete.

2. **An application.** We give an application of Theorem 1 to optimal control theory. Let B be a subset of R^n and C^* a Hausdorff space, and let $f^1, \cdots, f^n$ be continuous real-valued functions on $R \times B \times C^*$. An *admissible control function* is a measurable function $v: [a, b] \to C^*$, where $[a, b]$ is an interval in R; a *trajectory* corresponding to this control is an absolutely continuous function $x: [a, b] \to B$ such that

$$x^{i\prime}(t) = f^i(t, x(t), v(t)) \qquad (i = 1, \cdots, n)$$

for almost all t in $[a, b]$.

Two generalizations of this have been considered in optimal control theory and our application concerns the relation between them. For each (t, x) in $R \times B$, let $K(t, x)$ be the smallest convex set in R^n that contains the set

$$\{(f^1(t, x, v), \cdots, f^n(t, x, v)) : v \in C^*\}. \tag{2}$$

We almost, but not quite, follow J. Warga ([**7**], [**4**]) in defining a *relaxed admissible curve* to be an absolute continuous function $x: [a, b] \to B$ such that for almost all t in $[a, b]$

$$x'(t) \in K(t, x). \tag{3}$$

(Warga's definition has the closure of $K(t, x)$ in the right member of (3), which in the especially important case of compact C^* makes no difference.)

Let $\mathcal{P}$ be the set of probability measures defined on the σ-algebra of Borel subsets of C^*. A *relaxed control function* ([**5**], [**6**], [**9**]) is a function $(P_t: a \leqq t \leqq b)$ from $[a, b]$ to $\mathcal{P}$ such that for all bounded continuous functions $(\phi(t, v): a \leqq t \leqq b, v \in C^*)$ the function whose value at t is

$$\int_{C^*} \phi(t, v) P_t(dv) \tag{4}$$

is measurable on $[a, b]$. A *trajectory* corresponding to this relaxed control is an absolutely continuous function $x: [a, b] \to B$ such that for almost all t in $[a, b]$ the functions $(f^i(t, x(t), v): v \in C^*)$ are P_t-integrable over C^*, and

$$x^{i\prime}(t) = \int_{C^*} f^i(t, x(t), v) P_t(dv). \tag{5}$$

Since the point whose ith coordinate is the right member of (5) is in $K(t, x(t))$, every trajectory corresponding to a relaxed control is a relaxed admissible curve. We now prove a partial converse.

THEOREM 2. *If C^* is the union of a countable set $K_1, K_2, K_3, \cdots$ of compact metrizable sets, every relaxed admissible curve $(x(t): a \leqq t \leqq b)$ is a trajectory corresponding to a relaxed control function, and more specifically to a relaxed control function $(P_t: a \leqq t \leqq b)$ such that for each t there is a compact subset K_t of C^* for which $P_t(C^* - K_t) = 0$.*

This theorem generalizes Theorem 4.1 of [**7**].

For $q = 1, 2, \cdots$, let $\mathcal{P}_q$ be the set of those probability measures P on Borel subsets of C^* for which $P(C^* - K_q) = 0$. Then the union $\mathcal{P}_0$ of the $\mathcal{P}_q$ is contained in $\mathcal{P}$.

For each q there is a countable set $\{\phi_{q,1}, \phi_{q,2}, \cdots\}$ of continuous functions from K_q to R which is dense in the unit ball of the space $C(K_q)$ of all such functions [**3**, p. 245]. For each pair P', P'' of members of $\mathcal{P}_q$ we define

$$\rho_q(P', P'') = \sum_{j=1}^{\infty} 2^{-j} \left| \int_{K_q} \phi_{q,j}(v) P'(dv) - \int_{K_q} \phi_{q,j}(v) P''(dv) \right|.$$

This is a metric on $\mathcal{P}_q$. Convergence of $\rho_q(P^{(n)}, P^{(0)})$ to 0, $(P^{(n)}, P^{(0)} \in \mathcal{P}_q)$ is equivalent to convergence of $\int_{K_q} \phi(v) P^{(n)}(dv)$ to $\int_{K_q} \phi(v) P^{(0)}(dv)$ for all ϕ in the set $\{\phi_{q,1}, \phi_{q,2}, \cdots\}$, hence for all ϕ in $C(K_q)$. Given any sequence $P^{(1)}, P^{(2)}, \cdots$ in $\mathcal{P}_q$, by the diagonal process we can choose a subsequence (for which we retain the same notation) such that the limits

$$\lim_{n \to \infty} \int_{K_q} \phi_{q,j}(v) P^{(n)}(dv) \qquad (j = 1, 2, 3, \cdots)$$

exist. It follows at once that the limit

$$I(\phi) = \lim_{n\to\infty} \int_{K_q} \phi(v)P^{(n)}(dv)$$

exists for all ϕ in $C(K_q)$; it is linear, is nonnegative, if $\phi \geqq 0$, and is 1 if $\phi = 1$. Hence by the Riesz representation theorem there is a measure $P^{(0)}$ in $\mathcal{P}_q$ for which

$$I(\phi) = \int_{K_q} \phi(v)P^{(0)}(dv).$$

Then $\lim \rho_q(P^{(n)}, P^{(0)}) = 0$, so $\mathcal{P}_q$ is compact.

We shall need the following fact.

(6) If ϕ: $[a, b]\times K_q \to R$ is continuous, the function Φ_ϕ: $[a, b]\times P_q \to R$ whose value at (t, P) is $\int_{K_q}\phi(t, v)P(dv)$ is continuous.

Since ϕ is uniformly continuous, $\Phi_\phi(\tau, P')$ is uniformly continuous on $[a, b]$ for each fixed P' in $\mathcal{P}_q$. By definition of ρ_q, it is continuous in P' for each fixed t in $[a, b]$. Hence (6) holds.

Now we topologize $\mathcal{P}_0$ with the topology generated by the $\mathcal{P}_q$. A set $G \subseteq \mathcal{P}_0$ is open if and only if $G \cap \mathcal{P}_q$ is an open subset of $\mathcal{P}_q$ for $q = 1, 2, 3, \cdots$. Then a function on $\mathcal{P}_0$ is continuous if and only if its restriction to each $\mathcal{P}_q$ is continuous on $\mathcal{P}_q$. In particular, from (6) we obtain

(7) If ϕ: $[a, b]\times C^* \to R$ is continuous, the function Φ_ϕ: $[a, b] \times \mathcal{P}_0 \to R$ whose value at (t, P) $(t \in [a, b], P \in \mathcal{P}_0)$ is $\int_{C^*}\phi(t, v)P(dv)$ is continuous.

Let x: $[a, b]\to R$ be a relaxed admissible curve. There is a set M, consisting of almost all points of $[a, b]$, such that (3) holds for all t in M. We apply Theorem 1, letting A be R^{n+1} and Q be $[a, b]\times \mathcal{P}_0$, and defining k to be the function whose value at (t, P) is

$$k(t, P) = \left(t, \int_{C^*} f^1(t, x(t), v)P(dv), \cdots, \int_{C^*} f^n(t, x(t), v)P(dv)\right).$$

By (7), this is continuous on Q. For each t in M, the point $x'(t)$ of $K(t, x)$ is the weighted mean of finitely many points of the set (2), so that (5) holds with a P' concentrated on a finite subset of C^*, which is in $\mathcal{P}_q$ for all large q. So if we define

$$y(t) = (t, x^{1\prime}(t), \cdots, x^{n\prime}(t)) \qquad (t \in M)$$

we have $y(M) \subseteq k(Q)$. Clearly y is measurable, so by Theorem 1 there is a measurable function u: $M \to Q$ (whose value at t we denote by $(\tau(t), P_t)$) such that $k(u(t)) = y(t)$ on M; that is,

$$\tau(t) = t, \int_{C^*} f^i(t, x(t), v)P_t(dv) = x^{i\prime}(t) \qquad (t \in M, i = 1, \cdots, n).$$

We complete this by letting P_t be any measure in $\mathcal{P}_0$ on $[a, b]-M$.

If A is a closed subset of R, by (7) $\Phi_\phi^{-1}(A)$ is a closed subset of Q, hence is a countable union of compact sets. Therefore $u^{-1}(\Phi_\phi^{-1}(A))$ is a measurable set, and $\Phi_\phi \circ u$ is measurable. That is, (4) is measurable, so $(P_t: a \leqq t \leqq b)$ is a relaxed control function and x is a trajectory corresponding to it.

3. **A generalization.** If we permit the continuum hypothesis to be invoked, we can generalize Theorem 1 as follows.

THEOREM 4. *Let M be a measure space, A a Hausdorff space, and Q a separable metric space. Let $k: Q \rightarrow A$ be continuous, and $y: M \rightarrow A$ a measurable function such that $y(M) \subseteq k(Q)$. Then (assuming the continuum hypothesis) there is a measurable function $u: M \rightarrow Q$ such that $y = k \circ u$.*

Let $v_1, v_2, \cdots$ be a countable dense subset of Q. We map the set of closed subsets of Q into the set of all sequences of real numbers thus: to a closed set F in Q there corresponds the sequence $(d_1, d_2, \cdots)$ where d_n is the distance of v_n from F. This map is one-to-one and the cardinal of the set of sequences is the cardinal c of the continuum, so there are at most c closed subsets of Q.

By the continuum hypothesis, there is an ordinal Ω which is preceded by c ordinals, while if $\alpha < \Omega$, α is preceded by countably many ordinals. We can therefore label all compact subsets of Q with a subset of the ordinals less than Ω. Without loss of generality we may suppose that there is an ordinal $\Omega' \leqq \Omega$ and for every ordinal $\alpha < \Omega'$ a compact set C_α such that the C_α $(\alpha < \Omega')$ are all the compact subsets of Q.

Now for each $\alpha < \Omega'$ we define $Q_\alpha = \bigcup_{\beta \leqq \alpha} C_\beta$. This is the union of countably many compact sets. For each $\alpha < \Omega'$ define M_α to be the set of all $x \in M$ such that $y(x) \in k(Q_\alpha)$ and $y(x) \notin k(Q_\beta)$ for all $\beta < \alpha$. These sets are disjoint, and their union is M. Since Q_β and $\bigcup_{\beta<\alpha} C_\beta$ are countable unions of compact sets, M_α is measurable. By Theorem 1 there is a measurable function $u_\alpha: M_\alpha \rightarrow Q_\alpha$ such that $k(u_\alpha(x)) = y(x)$ for $x \in M_\alpha$. We define $u: M \rightarrow Q$ as follows: $u(x) = u_\alpha(x)$ if $x \in M_\alpha$. Since the M_α are disjoint, this is unambiguous.

We need only check that u is measurable. If K is a compact subset of Q then $K = C_\alpha$ for some α (fixed hereafter). Then $K \subseteq Q_\alpha$ and

$$u^{-1}(K) = \bigcup_{\beta \leqq \alpha} \overset{-1}{u_\beta}(K \cap Q_\beta).$$

Since the u_β are measurable functions and this is a countable union, it suffices to show that $K \cap Q_\beta$ is a countable union of compact sets for all $\beta \leqq \alpha$. This, however, is true since

$$K \cap Q_\beta = \bigcup_{j \leqq \beta} K \cap C_j.$$

This completes the proof of Theorem 4.

The proof given above actually applies to a larger class of spaces Q. Q could be any topological space with a family of compact metrizable subsets with cardinality at most c such that the union of the subsets in this family is all of Q and such that any compact subset of Q lies in the union of a countable subfamily.

Bibliography

1. A. F. Filippov, *On certain questions in the theory of optimal control*, Vestnik Moskov. Univ. Ser. Mat. Meh. Astronom. **2** (1959), 25–42; English transl., J. Soc. Indust. Appl. Math. Ser. A. Control **1** (1962), 76–84.

2. John G. Hocking and Gail S. Young, *Topology*, Addison-Wesley, Reading, Mass., 1961.

3. J. Kelley, *General topology*, Van Nostrand, Princeton, N. J., 1955.

4. Harold J. Kushner, *On the existence of optimal stochastic controls*, J. Soc. Indust. Appl. Math. Ser. A Control **3** (1966), 463–474.

5. E. J. McShane, *Generalized curves*, Duke Math. J. **6** (1940), 513–536.

6. ———, *Relaxed controls and variational problems*, J. Soc. Indust. Appl. Math. Ser. A Control (to appear).

7. J. Warga, *Relaxed variational problems*, J. Math. Anal. Appl. **4** (1962), 111–128.

8. ———, *Variational problems with unbounded controls*, J. Soc. Indust. Appl. Math. Ser. A Control **3** (1966), 424–438.

9. L. C. Young, *Generalized curves and the existence of an attained absolute minimum in the calculus of variations*, C. R. de la Soc. des Sci. et des Lettres de Varsovie, Classe III **30** (1937), 214–234.

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