A TWO-PARAMETER HOMOGENEOUS MEAN VALUE[1,2]

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1. Introduction. The mean of order t of the positive values $x=(x_1, x_2, \cdots, x_n)$ with positive weights $w=(w_1, w_2, \cdots, w_n)$, $\sum w_i=1$, is defined [3], [5] by

$$(1.1)\qquad \begin{aligned} M_t(x, w) &= \left[\sum_{i=1}^{n} w_i x_i^t\right]^{1/t}, \qquad t \neq 0, \\ M_0(x, w) &= \prod_{i=1}^{n} x_i^{w_i} = \lim_{t\to 0} M_t(x, w). \end{aligned}$$

Homogeneity in x distinguishes M_t from all other means of the form $\phi^{-1}[\sum_{i=1}^n w_i\phi(x_i)]$, where ϕ is any function with a unique inverse ϕ^{-1} [5, Theorem 84].

Without losing homogeneity, M_t has been generalized to the hypergeometric mean value $M(t, c; x; w)$ [4]. In the present paper we shall make a further generalization while maintaining homogeneity. We construct a two-parameter mean, $L(s, t; x)$, by first forming the mean $M_s(x, u)$. Using an arbitrary weight function, $P(u)$, we then take an integral average over all possible choices of the weights u satisfying $\sum_{i=1}^n u_i=1$. Because $u_n=1-u_1- \cdots -u_{n-1}$, the average requires an $(n-1)$-fold integration with respect to $u_1, u_2, \cdots, u_{n-1}$. If the variables x are all positive, then for any real s and t we define

$$(1.2)\qquad \begin{aligned} L(s, t; x) &= \left[\int_E M_s^t(x, u)P(u)du'\right]^{1/t}, \qquad t \neq 0, \\ L(s, 0; x) &= \lim_{t\to 0} L(s, t; x), \end{aligned}$$

where $u'=(u_1, \cdots, u_{n-1})$, $du'=du_1du_2 \cdots du_{n-1}$, $P(u)\geqq 0$, $\int_E P(u)du' =1$ and $E=\{u' \mid u_i>0,\ 1\leqq i\leqq n-1 \text{ and } u_n=1-u_1- \cdots -u_{n-1}>0\}$.

The $L(s, t; x)$ mean is homogeneous in x. It can be regarded as a special case of the integral mean [5, Chapter 3]

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$$M_t(f, P) = \left[\int f^t(u)P(u)du\right]^{1/t},$$

but it is a generalization of previously known ways of constructing a homogeneous mean of the discrete variables $x_1, x_2, \cdots, x_n$. After defining a set of natural weights w associated with the function $P(u)$, we shall show (Theorem 3) that $L(s, t; x)$ contains $M_t(x, w)$ in the special case $s=t$. It contains also the hypergeometric mean, $M(t, c; x; w)$, in the special case $s=1$ and $P(u)=P(cw; u)$, a particular weight function depending on the parameters $cw=(cw_1, cw_2, \cdots, cw_n)$ [**4**, Equation (2.2)]. All properties of $M(t, c; x; w)$ which do not depend explicitly on c can be generalized to properties of $L(s, t; x)$. In this generalization, s and t each play roles analogous to that of t in $M(t, c; x; w)$.

2. **Elementary properties of** $L(s, t; x)$. If $P(u)$ is such that $L^t(s, t; x)$ is an improper integral, it is easily shown to converge uniformly in s, t, and x for $0<m\leqq x_i\leqq M$, $1\leqq i\leqq n$, all real s, and $-T\leqq t\leqq T$. As a result of this uniform convergence and the continuity of $M_s^t(x, u)$ in s, t, x, and u, $L(s, t; x)$ is continuous in s, t, and x.

In considering some limiting values of $L(s, t; x)$, we use the notation $x_{\max}=\max\{x_1, x_2, \cdots, x_n\}$ and $x_{\min}=\min\{x_1, x_2, \cdots, x_n\}$.

THEOREM 1.

(a) $L(s, 0; x) = \lim_{t\to 0} L(s, t; x) = \exp\left[\int_E \ln M_s(x, u)P(u)du'\right]$,

(b) $\lim_{t\to\infty} L(s, t; x) = x_{\max}$, (c) $\lim_{t\to-\infty} L(s, t; x) = x_{\min}$,

(d) $\lim_{s\to\infty} L(s, t; x) = x_{\max}$, (e) $\lim_{s\to-\infty} L(s, t; x) = x_{\min}$.

PROOF. Part (a) is an application of L'Hôpital's rule, differentiation with respect to t under the integral sign being permissible because the integral of the derivative converges uniformly for $-T\leqq t\leqq T$ and $0<m\leqq x_i\leqq M$, $1\leqq i\leqq n$. Parts (b) and (c) follow from properties of the integral mean $M_t(f, p)$ [**5**, p. 143] with $f(u)=M_s(x, u)$. Parts (d) and (e) follow from properties of $M_s(x, u)$ [**5**, Theorem 4].

THEOREM 2. (a) *$L(s, t; x)$ is a strictly increasing function of x; i.e. if $x_i\leqq y_i$ for all i and $x_j<y_j$ for some j, then $L(s, t; x)<L(s, t; y)$.*

(b) *If $x_{\max}>x_{\min}$, then $L(s, t; x)$ is a strictly increasing function of t.*

(c) *If $x_{\max}>x_{\min}$, then $L(s, t; x)$ is a strictly increasing function of s.*

PROOF. (a) From the definition of $M_s(x, u)$ we see that $M_s(x, u)$

$<M_s(y, u)$. The result is evident by inspection of (1.2) and Theorem 1(a). Part (b) is a property of the integral mean $M_t(f, P)$ [**5**, p. 144] with $f(u) = M_s(x, u)$. For (c) it suffices to observe that $f = M_s(x, u)$ is a strictly increasing function of s [**5**, Theorem 16] and that $M_t(g, P) > M_t(f, P)$ if $g(u) > f(u)$ for all u.

The following theorem shows that the elementary mean $M_s(x, w)$ is a special case of the $L(s, t; x)$ mean; the weights $w = (w_1, w_2, \cdots, w_n)$ are the "natural weights" associated with the weight function $P(u)$. We define the natural weights by

$$w_i = \int_E u_i P(u) du', \quad 1 \leqq i \leqq n. \tag{2.1}$$

THEOREM 3. *If w denotes the natural weights, then* $L(s, s; x) = M_s(x, w)$.

PROOF. If $s \neq 0$,

$$L(s, s; x) = \left[\int_E M_s^s(x, u) P(u) du'\right]^{1/s} = \left[\int_E \sum_{i=1}^n u_i x_i^s P(u) du'\right]^{1/s}$$

$$= \left[\sum_{i=1}^n x_i^s \int_E u_i P(u) du'\right]^{1/s} = M_s(x, w).$$

If $s = 0$,

$$L(0, 0; x) = \exp\left[\int_E \ln \prod_{i=1}^n x_i^{u_i} P(u) du'\right]$$

$$= \exp\left[\sum_{i=1}^n \ln x_i \int_E u_i P(u) du'\right]$$

$$= \exp\left[\ln \prod_{i=1}^n x_i^{w_i}\right] = M_0(x, w).$$

A given set of weights w occurs as the natural weights associated with a large class of functions P. This class contains the family $P(cw; u)$ [**4**, Equation (2.2)] where the natural weights are just the parameters w. If $P(u) = P(cw; u)$, $L(s, t; x)$ becomes the "generalized hypergeometric mean" $L(s, t, c; x; w)$. In particular, $L(1, t, c; x; w) = M(t, c; x; w)$. For $s \neq 0$, $L(s, t, c; x; w)$ can be expressed in terms of $M(t, c; x; w)$ by using the identity

$$L^r(s, t; x) = \left[\int_E \left[\sum_{i=1}^n u_i (x_i^r)^{s/r}\right]^{(t/r)/(s/r)} P(u) du'\right]^{r/t} \tag{2.2}$$

$$= L(s/r, t/r; x^r), \quad r, s, t \neq 0.$$

A similar argument gives the same identity if s or t is zero. Putting $r=s$ and $P(u)=P(cw;\ u')$, we have

$$L(s,\ t,\ c;\ x;\ w)\ =\ [M(t/s,\ c;\ x^s;\ w)]^{1/s}.$$

Although Theorems 1 and 2 show at once that $x_{\min}<L(s,\ t;\ x)<x_{\max}$, the introduction of natural weights allows us to give sharper inequalities:

COROLLARY 1. *Let w denote the natural weights. If $x_{\max}>x_{\min}$ and $s<t$, then $M_s(x,\ w)<L(s,\ t;\ x)<M_t(x,\ w)$. The inequalities are reversed if $s>t$.*

PROOF. By Theorem 2, $L(s,\ t;\ x)$ is an increasing function of each of the parameters s and t. Hence, applying Theorem 3, if $s<t$,

$$M_s(x,\ w)\ =\ L(s,\ s;\ x)\ <\ L(s,\ t;\ x)\ <\ L(t,\ t;\ x)\ =\ M_t(x,\ w),$$

with reversed inequalities if $s>t$.

It is well known [1, p. 9] that if $\log f(r,\ u)$ is convex in r, then $\log \int f(r,\ u)du$ is convex in r. To study the convexity of $L^s(s,\ t;\ x)$ in s, we need the analogous theorem that if $r \log f(r,\ u)$ is convex in r, then $r \log \int f(r,\ u)du$ is convex in r for $r>0$.

LEMMA 1. *Let $f^r(r,\ u)$ be continuous in r and u and log convex in r. Then, if $[\int_E f(r,\ u)P(u)du']^r$ is continuous, it is log convex in r for $r>0$.*

PROOF. If $f^r(r,\ u)$ is continuous and log convex in r, then $[f^r(r,\ u)]^{1/r}=f(r,\ u)$ is continuous and log convex in the variable $1/r$ for $r>0$ [5, Theorem 119]. Since $f(r,\ u)$ is continuous in u and log convex in $1/r$, $\int_E f(r,\ u)P(u)du'$ is log convex in $1/r$ [1, p. 9]. Hence $[\int_E f(r,\ u)P(u)du']^r$ is log convex in r for $r>0$ [5, Theorem 119].

THEOREM 4. (a) *$L^s(s,\ t;\ x)$ is log convex in s for $t/s\geqq 0$.* (b) *$L^t(s,\ t\cdot\ x)$ is log convex in t.*

PROOF. (a) For any fixed $t\neq 0$, $L^s(s,\ t;\ x)=[\int_E M_s^t(x,\ u)P(u)du']^{s/t}=[\int_E M_r(y,\ u)P(u)du']^r$, where $y_i=x_i^t$, $1\leqq i\leqq n$, and $r=s/t$. Since $M_r^r(y,\ u)$ is log convex in r [5, Theorem 87], Lemma 1 implies $[\int_E M_r(y,\ u)P(u)du']^r$ is log convex in r for $r=s/t>0$. But if a function is log convex in r, it is log convex in $tr=s$ for any fixed $t\neq 0$, [1, Theorem 1.10].

For $t=0$, $\ln L^s(s,\ 0;\ x)=\int_E \ln M_s^s(x,\ u)P(u)du'$ is convex in s since $M_s^s(x,\ u)$ is log convex in s.

(b) $L^t(s,\ t;\ x)$ is log convex in t since $M_t^t(f,\ P)$ is log convex in t [5, Theorem 197], and $L^t(s,\ t;\ x)$ has this form with $f(u)=M_s(x,\ u)$.

3. **Inequalities for** $L(s,\ t;\ x)$. The next two theorems are results of

properties of the mean $M_s(x, u)$ and the integral mean $M_t(f, P)$. The first comes from Theorems 24, 186 and 198 of [5]. The second comes from Theorems 12 and 188 of [5], with the special case $t=0$ as an elementary result of properties of the logarithm. We use the notation $x+y=(x_1+y_1, x_2+y_2, \cdots, x_n+y_n)$ and $xy=(x_1y_1, x_2y_2, \cdots, x_ny_n)$.

THEOREM 5 (MINKOWSKI). *Let x and y be vectors with $x_i>0$ and $y_i>0$, $1 \leqq i \leqq n$. Then, unless $s=t=1$ or $x_i=ky_i$, $1 \leqq i \leqq n$,*

$$L(s, t; x + y) < L(s, t; x) + L(s,t; y), \qquad (s, t \geqq 1),$$

with reversed inequality if $s, t \leqq 1$. Equality holds in the exceptional cases.

THEOREM 6 (HÖLDER). *Let x and y be vectors with $x_i>0$ and $y_i>0$, $1 \leqq i \leqq n$, and let p and q be real numbers greater than unity such that $1/p+1/q=1$. Then, unless $s=t=0$ or $x_i^p=ky_i^q$, $1 \leqq i \leqq n$,*

$$L(s, t; xy) < L^{1/p}(s, t; x^p)L^{1/q}(s, t; y^q), \qquad (s, t \geqq 0),$$

with reversed inequality if $s, t \leqq 0$. Equality holds in the exceptional cases.

By defining the mean of a matrix of values x_{ij}, both of the preceding theorems can be included in an analogue of the Jessen-Ingham inequality [5, Theorems 26 and 203]. The proof [7, p. 20] relies primarily on the Minkowski inequality and uses Hölder's inequality for a special case.

Finally, we shall show that $L(s, t; x)$ satisfies a Kantorovich inequality [2, p. 208]. The proof proceeds by adapting a method due to Rennie [6, p. 982].

THEOREM 7 (RENNIE). *Let $0<A \leqq x_i \leqq B$, $1 \leqq i \leqq n$. Then if $t \neq 0$,*

$$L^t(s, t; x) + A^tB^tL^{-t}(s, -t; x) = L^t(s, t; x) + A^tB^tL^t(-s, t; 1/x) \leqq A^t + B^t,$$

with equality if and only if $x_i=A$ or $x_i=B$, $1 \leqq i \leqq n$.

PROOF. The equality between the first and second members is seen by (2.2) with $r=-1$. To obtain the inequality we notice that $M_s^t(x, u)$ is bounded between A^t and B^t and, following Rennie, we consider

$$[M_s^t(x, u) - A^t][1 - B^tM_s^{-t}(x, u)]P(u) \leqq 0, \ (t \neq 0).$$

Integrating and rearranging, we have

$$\int_E M_s^t(x, u)P(u)du' + A^tB^t \int M_s^{-t}(x, u)P(u)du' \leqq A^t + B^t.$$

Equality holds if and only if $M_s^t(x, u) = A^t$ or $M_s^t(x, u) = B^t$ for all $u \in E$.

THEOREM 8 (KANTOROVICH). *If* $t > 0$ *and* $0 < A \leqq x_i \leqq B$, $1 \leqq i \leqq n$, *then*

$$
\begin{aligned}
1 &\leqq L(s, t; x)/L(s, -t; x) = L(-s, t; x)L(s, t; 1/x) \\
&\leqq [(A^t + B^t)/2]^{1/t}[(A^{-t} + B^{-t})/2]^{1/t},
\end{aligned}
$$

with equality on the left if and only if $x_{\max} = x_{\min}$, *and equality on the right if and only if* $A = B$.

PROOF. The left inequality holds because $L(s, t; x)$ is a strictly increasing function of t by Theorem 2(b), unless $x_{\max} = x_{\min}$. The equality between the second and third members is due to (2.2) with $r = -1$. To obtain the right-hand inequality we start with Rennie's inequality (Theorem 7),

$$L^t(s, t; x) + A^t B^t L^{-t}(s, -t; x) \leqq A^t + B^t.$$

Dividing by 2, applying the inequality of the arithmetic and geometric means to the left side, and squaring, we find

$$
\begin{aligned}
[L(s, t; x)/L(s, -t; x)]^t &\leqq (A^t + B^t)^2/4A^t B^t \\
&= [(A^t + B^t)/2][(A^{-t} + B^{-t})/2].
\end{aligned}
$$

Taking the tth root gives the desired inequality. If $A = B$, we have equality at each step of the proof; if $A \neq B$, then the conditions for equality in Theorem 7 imply strict inequality of the arithmetic and geometric means.

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