ON FRACTIONAL INTEGRALS OF PURE IMAGINARY ORDER IN L_p

G. K. KALISCH[1]

The purpose of this paper is to extend some results of Kober [**4**] from L_2 to L_p, $1<p<\infty$. Sahnovič [**7**], using a similar method, obtained a result similar to Kober's (in L_2). His purpose was to show the similarity of the operators M and $M+iV$ in $L_2(0, 1)$ where $(Mf)(x)=xf(x)$ and $(Vf)(x)=\int_0^x f(y)\,dy$; he found that the operator P implementing this similarity, that is, a bounded operator P with bounded inverse P^{-1} such that $P^{-1}MP=M+iV$, is precisely $P=J^i$ where J^ζ is the operator of fractional integration

$$(J^\zeta f)(s) = (1/\Gamma(\zeta)) \int_0^s (s-t)^{\zeta-1} f(t)\,dt$$

where $\zeta=\xi+i\eta$ is a complex variable with real ξ and η and f and $J^\zeta f$ (if it exists) are functions defined on the interval $[0, 1]$. Our motivation is similar to Sahnovič's; we are publishing elsewhere [**3**] the application of our present results to the type of problem considered by him except that we omit the restriction $p=2$. We are using without further reference results and notations from the exposition by Hille and Phillips [**2**] of Kober's results such as the boundedness and the strongly continuous holomorphic semigroup character of J^ζ for $\xi>0$.

For this semigroup we prove the conclusions of Theorems 17.9.1 and 17.9.2 in Hille and Phillips.

THEOREM G. *The holomorphic semigroup $\{J^\zeta\}$ with $\xi>0$ defined in $L_p=L_p(0, 1)$ with $1<p<\infty$ admits a bounded boundary group $\{J^{i\eta}\}$ in the sense that the L_p-limit*

$$\lim_{\xi\to 0+} J^{\xi+i\eta} f = J^{i\eta} f$$

exists for all $f\in L_p$. The set $\{J^{i\beta}\}$ for all real β is a strongly continuous group of bounded operators, $J^{i\beta}$ commutes with J^ζ for all $\xi>0$ and all real β and η and

$$J^{\xi+i\eta} = J^\xi J^{i\eta}.$$

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If A is the infinitesimal generator of $\{J^{\xi}\}$ then iA is the infinitesimal generator of $\{J^{i\beta}\}$.

The proof will be accomplished if the hypothesis of theorems 17.9.1 and 17.9.2 in Hille and Phillips is verified. This is done in the following theorem.

THEOREM B. *Let $\xi \in (0, 1]$ and $\eta \in [-1, 1]$. Then there exists a finite positive constant M independent of ξ and η but dependent on $p \in (1, \infty)$ such that the L_p-norm of J^{ζ} satisfies*

$$\|J^{\zeta}\|_p < M.$$

The proof is similar to the proofs by Kober and Sahnovič; the missing ingredient is Mihlin's extension to Fourier integrals in L_p [6] of Marcinkiewicz's result on multipliers of Fourier series in L_p [5]; see also Zygmund [8].

Write

$$g(s) = (J^{\zeta} f)(s) = (d/ds)(J^{\zeta+1} f)(s)$$
$$= \frac{1}{\Gamma(\zeta + 1)} \frac{d}{ds}\left(\int_0^s (s - t)^{\zeta} f(t)\, dt\right)$$

for $f \in L_p(0, 1)$ and $\xi > 0$. We shall use the following substitutions: $s = e^{\sigma}$, $t = e^{\tau}$, $\phi(\tau) = e^{(\tau/p)} f(e^{\tau})$, $\gamma(\sigma) = e^{(\sigma/p)} g(e^{\sigma})$. If f and g are in $L_p(0, 1)$, then ϕ and γ are in $L_p(-\infty, 0)$ and conversely, and we have $\|f\|_p = \|\phi\|_p$ and $\|g\|_p = \|\gamma\|_p$. We then obtain

$$\gamma(\sigma) = \frac{e^{-a\sigma}}{\Gamma(\zeta + 1)} \frac{d}{d\sigma}\left(\int_{-\infty}^{\sigma} (e^{\sigma} - e^{\tau})^{\zeta} \phi(\tau)\, e^{a\tau}\, d\tau\right) \tag{1}$$

where $(1/p) + a = 1$. Write

$$\phi(\tau) = \left(\frac{1}{(2\pi)^{1/2}}\right) \int_{-\infty}^{\infty} e^{-i\tau\beta}\, \Phi(\beta)\, d\beta. \tag{2}$$

We now confine ourselves to a set of functions f such that (2) is valid with corresponding Φ in $L_1(-\infty, \infty)$; this set is dense in $L_p(0, 1)$. After substituting (2) into (1) and interchanging the order of integration, we can evaluate the inner integral and obtain

$$\gamma(\sigma) = e^{-a\sigma} \frac{d}{d\sigma}\left[\left(\frac{1}{(2\pi)^{1/2}}\right) \int_{-\infty}^{\infty} e^{\sigma(\zeta + a - i\beta)} \frac{\Gamma(a - i\beta)}{\Gamma(\zeta + 1 + a - i\beta)}\, \Phi(\beta)\, d\beta\right]$$
$$= \left(\frac{e^{2\sigma\zeta}}{(2\pi)^{1/2}}\right) \int_{-\infty}^{\infty} e^{-i\sigma\beta} \frac{\Gamma(a - i\beta)}{\Gamma(\zeta + a - i\beta)}\, \Phi(\beta)\, d\beta.$$

We show below that $\Psi(\beta)=(\Gamma(a-i\beta)/\Gamma(\zeta+a-i\beta))$ is bounded in absolute value uniformly with respect to ζ in the range considered in the theorem; this shows that our hypothesis on Φ allows us to differentiate under the integral sign. We now restrict f again so that $\phi\in L_1(-\infty, 0)$; this still leaves us with a set of f's dense in $L_p(0, 1)$. This last restriction allows us to write

$$\Phi(\beta) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} e^{i\alpha\beta}\phi(\alpha)\,d\alpha$$

so that

$$\gamma(\sigma) = e^{\sigma\zeta}\int_{-\infty}^{\infty} e^{-i\beta\sigma}\Psi(\beta)\int_{-\infty}^{\infty} e^{i\alpha\beta}\phi(\alpha)\,d\alpha d\beta.$$

If now there exists a constant M independent of β and ζ (in our range) and ϕ such that

$$(3) \qquad |\Psi(\beta)| < M, \qquad |\beta\Psi'(\beta)| < M$$

for all real β then we can apply Mihlin's theorem and conclude that $\|\gamma\|_p < M\|\phi\|_p$ or $\|g\|_p = \|J^\zeta f\|_p < M\|f\|_p$ which is the desired conclusion for a set of f's dense in $L_p(0, 1)$ and hence for all f in $L_p(0, 1)$.

The two inequalities (3) are simple consequences of Stirling's formula for $\Gamma(z)$ and $\log\Gamma(z)$ as found for example in Franklin [1] since it is clearly sufficient to check them for large $|\beta|$.

Thus for the first inequality (3), we have the asymptotic relation for large $|\beta|$

$$|\Psi(\beta)| \sim \exp(\xi)\ |\beta|^{-\xi}\exp[\eta\arg(\zeta+a-i\beta)]$$
$$\cdot\exp(-\beta[\arg(a-i\beta)-\arg(\zeta+a-i\beta)]).$$

The angles arg . . . are comprised between $-\pi/2$ and $\pi/2$. On checking the right side of the above asymptotic relation, we see that it is bounded by $\exp(2+\pi/2)$ and that its limit as $|\beta|\to\infty$ is zero.

For the second inequality (3), we have

$$\beta\Psi'(\beta)=\Psi(\beta)\cdot\beta[((d/dz)\log\Gamma)(a-i\beta)-((d/dz)\log\Gamma)(\zeta+a-i\beta)].$$

We know that the first factor is bounded in absolute value. The boundedness of the second factor follows from the formula [1]

$$\log\Gamma(z) = -z+(z-\tfrac{1}{2})\log z+\log(2\pi)^{1/2}+\left(\frac{1}{12z}\right)-\int_0^{\infty}\frac{P_2(x)}{(z+x)^2}\,dx.$$

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University of Minnesota and
University of California, Irvine