ON THE SOLUTIONS OF ABSTRACT NONLINEAR EQUATIONS[1]

WALTER A. STRAUSS

1. We collect here some results on the existence of solutions to the equation $f(u)=0$ where f is a function from a Banach space to its dual. Hypotheses of continuity and of coercivity (positivity) are required. Here we discuss in detail only the latter. Our main new point is that it can be weakened in the Hilbert space case (Theorem 3). More restrictive in three important respects are the results of Shinbrot [6].

In the finite-dimensional case, the Brouwer fixed point theorem is immediately applicable, although various more involved ideas have been used in the past (cf. Višik [8], Shinbrot [6]). (I am indebted to H. Samelson for some enlightening discussions in this respect.) The general case of the equation $f(u)=0$, where $f: X \to Y$, is then obtained as a limit of solutions of $p_F f j_F(u)=0$ where F is a finite-dimensional subspace of X, j_F is the inclusion of F into X, and $p_F: Y \to F$. This includes the well-known Galerkin approximation (cf., for instance, Hopf [2], Browder [1]). We take Y as the dual of X, which has generally been the most useful case, although the case $X=Y$ has been considered by Kaniel [3].

2. **Finite-dimensional case.** Let B denote the closed unit ball in n-space and S its boundary.

LEMMA 1. *Any nonvanishing (continuous) vector field on B must point toward the origin at some point of S.*

This well-known fact is a simple variant of Brouwer's theorem. Indeed, such a vector field may be normalized to obtain a mapping f from B to S. Then apply Brouwer's theorem to $-f$. Conversely, if Lemma 1 holds and g maps B into B, define $h(u)=u-g(u)$, $u \in B$. It is clear that h represents a vector field which nowhere on S points toward the origin. Hence h vanishes somewhere on B and g has a fixed point.

The next lemma is a variation due to Browder [1].

LEMMA 2. *Let B now denote the closed unit ball in complex n-space, and $S=\partial B$. Let $n>1$. For any nonvanishing (continuous) vector field on B, there exists a point $v \in S$ where the vector field is orthogonal to both v and iv $(i=(-1)^{1/2})$.*

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PROOF. Normalize the given vector field to obtain a map f from B to S. Let $(\ ,\ )$ denote the complex inner product. If the conclusion were false, define

$$\alpha(u) = (f(u), u)^{-}/|(f(u), u)| \qquad (u \in S),$$

so that α maps S into the circle C in the complex plane. But the homotopy group $\pi_{2n-1}(C)$ is trivial, so that α must be homotopic to a constant. This implies that α has a continuous extension α^* from B to C. The product $g(u) = \alpha^*(u)f(u)$ then maps B into S; by Lemma 1 there exists $v \in S$ such that $g(v) = -v$. Hence $|(f(v), v)| = -1$, which is absurd.

Concerning an arbitrary norm, we have

LEMMA 3. *Let X be a finite-dimensional (real or complex) normed linear space and X' its dual space; the pairing between X and X' is denoted by $(\ ,\ )$. Let $B = \{u \in X \mid |u|_X \leqq r\}$ and $S = \partial B$. If a mapping f from B to X' vanishes nowhere, then*

(i) *there exists $v \in S$ with* $\mathrm{Re}(fv, v) < 0$,

(ii) *there exists $w \in S$ with $(fw, w) = 0$ provided that X is not a complex one-dimensional space.*

PROOF. First assume X is real. Then (ii) follows from (i) by the continuity of f. Let E_n denote n-space provided with the usual Euclidean norm; E_n may be identified with its own dual. By choosing a basis of X, we find a linear homeomorphism L from E_n onto X. Let L^* be its adjoint and

$$Ju = \{|u|/|Lu|_X\}u \qquad (0 \neq u \in E_n),$$

$J0 = 0$. Then LJ is an isometry for E_n to X, so that $f_1 = L^*fLJ$ defines a nonvanishing vector field on the unit ball in E_n. By Lemma 1, $f_1v_1 = -v_1$ for some point v_1 on the unit sphere. Then $v = LJv_1$ satisfies the desired conditions.

Now let X be complex. By considering X as a real space of twice the number of dimensions, (i) follows. To prove (ii), define f_1 as before with E_n replaced by complex n-space. By Lemma 2, $(f_1w_1, w_1) = 0$ for some point w_1 on the unit sphere. Let $w = LJw_1$.

3. Passage to the limit.

THEOREM 1. *Let X be a Banach space, the dual of another Banach space. Let f be a function from the ball $B = \{u \in X \mid |u| \leqq r\}$ in X to a bounded subset of X'. Assume*

(I) *(fu, v) is a continuous function of u, whenever u and v are restricted to a finite-dimensional subset.*

(II) *f is also continuous in the following sense: whenever a net $\{u_\alpha\}$ in X converges weakly* to u, and the net $\{fu_\alpha\}$ in X' converges weakly* to zero and $(fu_\alpha, u_\beta)=0$ for $\alpha \geqq \beta$, then we have $fu=0$.*

(III)$_1$ $\mathrm{Re}(fu, u) \geqq 0$ *for all* $u \in X$, $|u| = r$.

Then f sends some element of B into zero.

THEOREM 2. *Same, but with* (III)$_1$ *replaced by*:

(III)$_2$ $(fu, u) \neq 0$ *for all* $u \in X$, $|u| = r$; *X not a complex one-dimensional space.*

THEOREM 3. *Same, but with* (III)$_1$ *replaced by*:

(III)$_3$ *X is a Hilbert space and* $fu + \lambda u \neq 0$ *for all* $u \in X$, $|u| = r$, *for all* $\lambda > 0$.

PROOFS. For any finite-dimensional subspace F of X, furnished with the induced norm, let j_F denote the inclusion of F into X and j'_F its adjoint from X' to F'. By (I), $j'_F f j_F$ (restricted to the ball of radius r) is continuous from $F \cap B$ to F'. In case of (III)$_1$ or (III)$_2$, Lemma 3 shows that $j'_F f j_F$ sends some element $u_F(|u_F| \leqq r)$ into zero.

In case X is a Hilbert space, there is a real-linear isometry of F onto a Euclidean space. (In the complex case, F is considered as a real Hilbert space of twice the dimension.) Thus Lemma 1 implies that $j'_F f j_F$ sends some element of $B \cap F$ into zero. So in all three cases we obtain a net $\{u_F\}$ where the set of finite-dimensional subspaces F of X is directed by inclusion.

Bounded sets being weakly*-compact, we may extract a subnet $\{u_\alpha\}$ which converges weakly* to some element u of X and such that $\{fu_\alpha\}$ converges weakly* to some $v \in X'$. It follows that v annihilates every F; hence $v=0$. The assumptions of (II) are therefore satisfied and we conclude that $fu=0$.

REMARK 1. If we apply Theorem 1 or 2 to the operators $u \rightarrow fu - v$, where $v \in X'$ and $f\colon X \rightarrow X'$, we obtain known criteria for f to map onto X'.[2] Specifically, (III)$_2$ holds for every such operator provided $\dim X > 1$ and

$$|(fu, u)|/|u| \rightarrow \infty \quad \text{as} \quad |u| \rightarrow \infty.^{3}$$

REMARK 2. The crucial hypothesis (II) automatically holds if X is

[2] Of course, corresponding fixed point theorems are obtained by applying the present results to the mapping $u \rightarrow fu - u$.

[3] An example included in Theorem 3 but not Theorems 1 and 2, is the trivial functional equation $\phi(u(x)) = v(x)$ if ϕ is a bounded continuous function such that $|\arg \phi(z) - \arg z| < \pi$ for all z but $> \pi/2$ for some z.

reflexive[4] and f is *monotonic*; that is,

$$\operatorname{Re}(fu - fv, u - v) \geqq 0 \qquad (u, v \in B)$$

(cf. Minty [5]). For a useful and more general situation, see Leray and Lions [4]. For two different cases when (II) holds, see Strauss [7, §6] and Browder [1]; in these cases, the approximate solutions u_F actually converge *strongly* in X to the solution.

Bibliography

1. F. E. Browder, *Remarks on nonlinear functional equations*, Illinois J. Math. **9** (1965), 608–616.

2. E. Hopf, *Über die Aufangswertaufgabe für die hydrodynamischen Grundgleichungen*, Math. Nachr. **4** (1951), 213–231.

3. S. Kaniel, *Quasi-compact operators*, (to appear).

4. J. Leray and J. L. Lions, *Quelques résultats de Višik sur les problèmes elliptiques nonlinéaires par les méthodes de Minty-Browder*, Bull. Soc. Math. France **93** (1965), 97–107.

5. G. J. Minty, *Monotone (nonlinear) operators in Hilbert space*, Duke Math. J. **29** (1962), 341–346.

6. M. Shinbrot, *A fixed point theorem and some applications*, Arch. Rational Mech. Anal. **17** (1964), 255–271.

7. W. A. Strauss, *Evolution equations nonlinear in the time-derivative*, J. Math. Mech. **15** (1966), 49–82.

8. M. I. Višik, *Solution of a system of quasi-linear equations in divergence form under periodic boundary conditions*, Dokl. Akad. Nauk **137** (1961), 502–505 = Soviet Math. Dokl. **2** (1961), 293–297.

Brown University

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[4] More generally, X may be the dual of another Banach space Y if f maps X into Y (considering $Y \subset Y''$). Cf. Abstract 66T-139, Notices Amer. Math. Soc. **13** (1966), 249.